Drinfeld type presentations for twisted Yangians

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Outline

- Yangians
  - R-matrix presentation
  - Drinfeld presentation
- Twisted Yangians
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  - Drinfeld type presentation

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R-matrix presentation for Yangians

- $V = \mathbb{C}^N$
- $R(u) := I - Pu^{-1} \in \mathrm{End}(V \otimes V)[u^{-1}], \qquad P : x \otimes y \mapsto y \otimes x.$
- Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v).$$

- [Faddeev-Reshetikhin-Takhtajan] Yangian $Y(\mathfrak{gl}_N)$ is $\mathbb{C}$-algebra with generators $t_{ij}^{(r)}, r > 0, 1 \leq i, j \leq N$ subject to

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v).$$

$$T(u) = \left(t_{ij}(u)\right)_{1\leq i,j\leq N}, \qquad t_{ij}(u) = \delta_{ij} + \sum_{r>0} t_{ij}^{(r)} u^{-r}.$$

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- $Y(\mathfrak{sl}_N) := Y(\mathfrak{gl}_N)$/center.
- $Y(\mathfrak{g})$ can be defined using R-matrix presentation for $\mathfrak{g}$ classical type
- $Y(\mathfrak{g})$ is a Hopf algebra deformation of $\mathfrak{g}[u]$. $t_{ij}^{(r)} \rightsquigarrow E_{ij}u^r$.
- coproduct $\Delta : T(u) \mapsto T(u) \otimes T(u), t_{ij}(u) \mapsto \sum_k t_{ik}(u) \otimes t_{kj}(u)$.

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Drinfeld presentation for Yangians

- Cartan matrix $C = (c_{ij})$ and Chevalley generators $h_i, e_i, f_i, i \in \mathbb{I}$ for $\mathfrak{g}$
- [Drinfeld'88] $Y(\mathfrak{g})$ is generated by $\kappa_{i,r}, \xi^{\pm}_{i,r}, i \in \mathbb{I}, r \in \mathbb{N}$, subject to

$$[\kappa_{i,r}, \kappa_{j,s}] = 0, \qquad [\kappa_{i,0}, \xi^{\pm}_{j,s}] = \pm c_{ij}\xi^{\pm}_{j,s}, \qquad [\xi^{+}_{i,r}, \xi^{-}_{j,s}] = \delta_{ij}\kappa_{i,r+s},$$

$$[\kappa_{i,r+1}, \xi^{\pm}_{j,s}] - [\kappa_{i,r}, \xi^{\pm}_{j,s+1}] = \pm\frac{c_{ij}}{2}\{\kappa_{i,r}, \xi^{\pm}_{j,s}\},$$

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$$\mathrm{Sym}_{r_1,\cdots,r_n}[\xi^{\pm}_{i,r_1}, [\xi^{\pm}_{i,r_2}, \cdots [\xi^{\pm}_{i,r_n}, \xi^{\pm}_{j,s}] \cdots]] = 0, \qquad i \neq j, n = 1 - c_{ij}.$$

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Isomorphism for two presentations

- Gauss decomposition

$$T(u) = \begin{pmatrix} 1 & & \\ F_{21}(u) & 1 & \\ F_{31}(u) & F_{32}(u) & 1 \end{pmatrix} \begin{pmatrix} D_1(u) & & \\ & D_2(u) & \\ & & D_3(u) \end{pmatrix} \begin{pmatrix} 1 & E_{12}(u) & E_{13}(u) \\ & 1 & E_{23}(u) \\ & & 1 \end{pmatrix}$$

- $\xi_i^+(u) = \sum_{k\geq 0} \xi_{i,r}^+ u^{-r-1} = E_{i,i+1}(u - \frac{i-1}{2})$.
- $\kappa_i(u) = \sum_{k\geq 0} \kappa_{i,r} u^{-r-1} = 1 - D_i(u - \frac{i-1}{2})^{-1} D_{i+1}(u - \frac{i-1}{2})$.
- [Drinfeld'88, Brundan-Kleshchev'05] Isomorphism for type A
  [Jing-Liu-Molev'18, Guay-Regelskis-Wendlandt'18] Isomorphism for type BCD

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Symmetric pair

- θ : involution on $\mathfrak{g}$.
- $(\mathfrak{g}, \mathfrak{g}^{\theta})$ symmetric pair.

Example

AI: $(\mathfrak{sl}_N, \mathfrak{so}_N)$. AII: $(\mathfrak{sl}_{2m}, \mathfrak{sp}_{2m})$. AIII: $(\mathfrak{sl}_{p+q}, \mathfrak{sl}_p \oplus \mathfrak{gl}_q)$.

- Lift θ to involution $\check{\theta}$ on $\mathfrak{g}[u]$ by $\check{\theta} : xu^r \mapsto (-1)^r \theta(x) u^r, x \in \mathfrak{g}$.
- Twisted Yangian is a coideal subalgebra of Yangians as a deformation of $\mathfrak{g}[u]^{\check{\theta}}$.
- [Olshanski'92]: AI, AII [Molev-Ragoucy'02]: AIII [Guay-Regelskis'16]: BCD
- Twisted Yangians provide solutions of reflection equations, which are crucial in quantum integrable system with boundaries conditions [Sklyanin'88].

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R-matrix presentation for twisted Yangian

- Y^{tw} of type AI is the subalgebra of $Y(\mathfrak{sl}_N)$ generated by $s_{ij}^{(r)}, r > 0, 1 \leq i, j \leq N$,

$$S(u) = (s_{ij}(u)) := T(u)T^t(-u),$$
$$s_{ij}(u) = \delta_{ij} + \sum_{r>0} s_{ij}^{(r)} u^{-r}$$

- Y^{tw} of type AI is generated by $s_{ij}^{(r)}$ subject to

$$R(u-v)S_1(u)R^t(-u-v)S_2(v) = S_2(v)R^t(-u-v)S_1(u)R(u-v),$$
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Construct a Drinfeld type presentation for twisted Yangians of split type.

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Twisted current algebra

- $\omega : e_i \mapsto -f_i, f_i \mapsto -e_i, h_i \mapsto -h_i$.
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. $\mathfrak{k} = \mathfrak{g}^\theta$. $\omega|_\mathfrak{p} = -1$.
- Twisted current algebra $\mathfrak{g}[u]^{\check{\omega}} = \mathfrak{k} \oplus \mathfrak{p}u \oplus \mathfrak{k}u^2 \oplus \mathfrak{p}u^3 \cdots$
- $\mathfrak{g}[u]^{\check{\omega}}$ is generated by

$$t_{i,s} = \big((-1)^s - 1\big)h_i u^s, \qquad b_{i,r} := \big((-1)^r f_i - e_i\big)u^r, \qquad i \in \mathbb{I}, r, s \in \mathbb{N}, s \text{ odd}.$$

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Drinfeld presentation for twisted Yangian

[Lu-Wang-Z'23] Let $C=(c_{ij})_{i,j\in\mathbb{I}}$ be simply-laced Cartan matrix. Define $\mathbb{C}$-algebra $\mathcal{Y}=\mathcal{Y}(C)$ with generators $h_{i,s}$, $b_{i,r}$, for $i\in\mathbb{I}, r,s\in\mathbb{N}$ and relations

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Theorem (Lu-Wang-Z'23)

Let C be Cartan matrix for $\mathfrak{sl}_N$. Then $\mathcal{Y}(C)$ is isomorphic to Y^{tw} assoc. $(\mathfrak{sl}_N, \mathfrak{so}_N)$.

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- [Lu-Wang-Z'24] We have a formulation for $\mathcal{Y}$ of arbitrary split type.
  - We expect that $\mathcal{Y}$ for split type BCD are isomorphic to twisted Yangians Y^{tw} in R-matrix presentation.
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Affine quantum groups and $\imath$quantum groups

- $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c$ untwisted affine Lie algebra
- Lift θ to an involution $\widehat{\theta}$ on $\widehat{\mathfrak{g}}$ by $\widehat{\theta} : xt^k \mapsto \theta(x)t^{-k}, c \mapsto -c$.

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Applications

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