Universal polynomial $\mathfrak{so}$ and $\mathfrak{q}$ weight systems

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Joint work with

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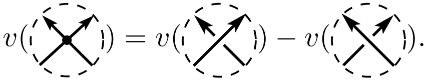
Finite type invariants

Any knot invariant v can be extended to knots with double points by means of the Vassiliev skein relation:



A knot invariant is said to be a Vassiliev invariant (or a finite type invariant) of order (or degree) $\leq n$ if its extension vanishes on all singular knots with more than n double points.
We shall denote by $\mathcal{V}_n$ the set of Vassiliev invariants of order $\leq n$ with values in a ring R and $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$, so we have an increasing filtration

$$\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_n \subseteq \cdots \subseteq \mathcal{V} := \bigcup_{n=0}^{\infty} \mathcal{V}_n \ .$$

Chord Diagram

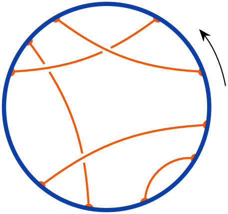


Figure: Chord Diagram

A chord diagram D of order n is an oriented circle with a set of n disjoint pairs of distinct points, considered up to orientation preserving diffeomorphisms of the circle. We denote the set of chords of a chord diagram D by $[D]$.

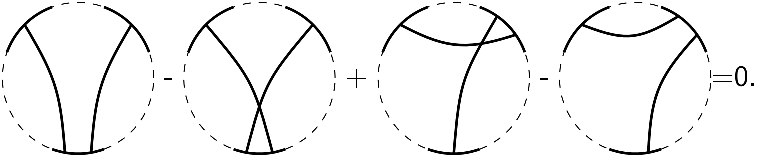
The vector space $\mathcal{A}$ spanned by chord diagrams over complex number $\mathbb{C}$ is graded,

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3 \oplus \dots .$$

Each component $\mathcal{A}_n$ is spanned by diagrams of the same order n.

4-term element

A 4-term (or $4T$) element is the alternating sum of the following quadruple of diagrams:



Here all the four chord diagrams contain, in addition to the two depicted chords, one and the same set of other chords.

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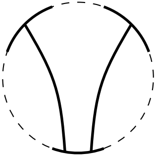
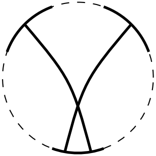
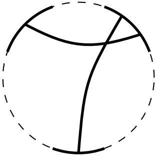
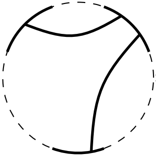
 -  $+$  -  $=0$.

Here all the four chord diagrams contain, in addition to the two depicted chords, one and the same set of other chords.

Each Vassiliev knot invariant of order at most n determines a function f on chord diagrams with n chords that vanishes on all 4-term elements. Such a function f is called a weight system.

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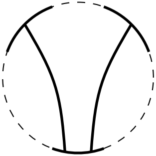
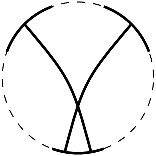
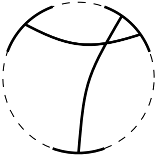
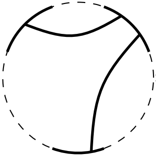
 -  +  -  $=0$.

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Each Vassiliev knot invariant of order at most n determines a function f on chord diagrams with n chords that vanishes on all 4-term elements. Such a function f is called a weight system.
According to (Kontsevich, 1993), every weight system arises from a certain Vassiliev invariant.

Lie algebra weight systems

$\mathfrak{g}$ is a metrized Lie algebra over $\mathbb{C}$ with an ad-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$,

- choose a basis $e_1, \ldots, e_d$, d is the dimension of $\mathfrak{g}$;
- dual basis $e_1^*, \ldots, e_d^*$ with respect to the form $\langle \cdot, \cdot \rangle$.

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Given a chord diagram D with n chords, choose a base point on the circle, away from the ends of the chords of D. This gives a linear order on the endpoints of the chords, increasing in the positive direction of the circle.

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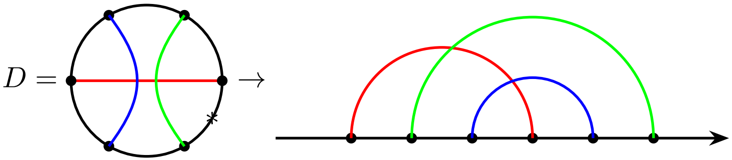
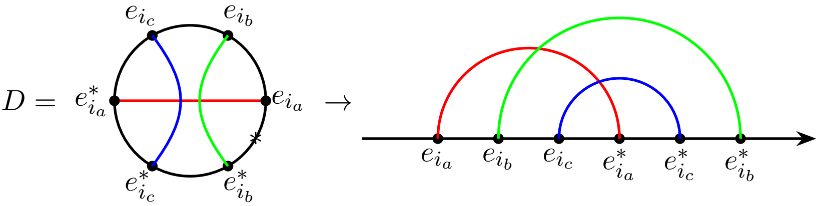
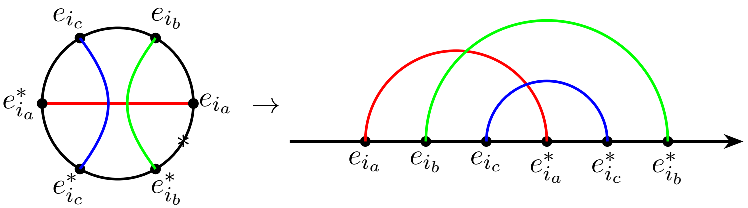


Lie algebra weight systems

Assign to each chord a an index, that is, an integer-valued variable, i_a. The values of i_a will range from 1 to d, the dimension of the Lie algebra. Mark the first endpoint of the chord with the symbol e_{i_a} and the second endpoint with $e^*_{i_a}$.



Lie algebra weight systems

$D =$ 

Write the product of all the e_{i_a} and all the $e^*_{i_a}$, in the order in which they appear on the circle of D, and take the sum of the d^n elements of the universal enveloping algebra $U(\mathfrak{g})$ obtained by substituting all possible values of the indexes i_a into this product. Denote by $w_{\mathfrak{g}}(D)$ the resulting element of $U(\mathfrak{g})$.

$$w_{\mathfrak{g}}(D) = \sum_{i_a=1}^{d} \sum_{i_b=1}^{d} \sum_{i_c=1}^{d} e_{i_a} e_{i_b} e_{i_c} e^*_{i_a} e^*_{i_c} e^*_{i_b} \in U(\mathfrak{g}).$$

Lie algebra weight systems

Theorem (D. Bar-Natan, M. Kontsevich, 1990s)

The above construction has the following properties:

1. *the element $w_{\mathfrak{g}}(D)$ does not depend on the choice of the base point on the diagram;*
2. *it does not depend on the choice of the basis e_i of the Lie algebra;*
3. *it belongs to the ad-invariant subspace*

$$U(\mathfrak{g})^{\mathfrak{g}} = \{x \in U(\mathfrak{g}) | xy = yx \text{ for all } y \in \mathfrak{g}\} = ZU(\mathfrak{g}).$$

4. *This map $w_{\mathfrak{g}}$ from chord diagrams to $ZU(\mathfrak{g})$ satisfies the 4-term relations. Therefore, it extends to a homomorphism of commutative algebras $w_{\mathfrak{g}} \colon \mathcal{A}^{fr} \to ZU(\mathfrak{g})$.*

This construction is easy to implement, hard to compute.

$\mathfrak{gl}(N)$ weight system

The Lie algebra $\mathfrak{gl}(N)$ consists of all $N \times N$ matrices,

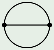
- the trace of the product of matrices as the preferred ad-invariant form: $\langle x, y \rangle = \mathrm{Tr}(xy)$;
- basis E_{ij} having 1 on the intersection of i th row with j th column and 0, $i, j = 1, \dots, N$;
- dual basis $E_{ij}^* = E_{ji}$, because of $\langle E_{ij}, E_{kl} \rangle = \delta_{il}\delta_{jk}$.

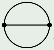
The commutation relations

$$[E_{kl}, E_{ji}] = E_{kl}E_{ji} - E_{ji}E_{kl} = \delta_{lj}E_{ki} - \delta_{ik}E_{jl}.$$

Casimir element

Example

For a chord diagram  with a single chord, we have

$w_{\mathfrak{gl}(N)}($  $) = \sum_{i,j=1}^{N} E_{ij}E_{ji}.$

We denote this element by $C_2 \in ZU(\mathfrak{gl}(N))$ and call the second Casimir. Similarly, $\sum_{i=1}^{N} E_{ii} = C_1$, and, more generally,

$$C_k = \sum_{i_1,i_2,\ldots,i_k=1}^{N} E_{i_1i_2}E_{i_2i_3}\ldots E_{i_ki_1}$$

is the k th Casimir element in $ZU(\mathfrak{gl}(N))$.

The center $ZU(\mathfrak{gl}(N))$ is isomorphic to the ring of polynomials in the Casimir elements $C_1, \ldots, C_N$. The higher Casimir elements $C_{N+1}, C_{N+2}, \ldots$ can be expressed as polynomials in $C_1, \ldots, C_N$.

$\mathfrak{gl}(N)$ weight system for permutations

Definition

For a permutation $\sigma \in S_m$, set

$$w_{\mathfrak{gl}(N)}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_{\sigma(1)}} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} \in U(\mathfrak{gl}(N)).$$

Claim

- For any σ, $w_{\mathfrak{gl}(N)}(\sigma)$ lies in the center of $U(\mathfrak{gl}(N))$.
- This element is invariant under conjugation by the standard cyclic permutation: $w_{\mathfrak{gl}(N)}(\sigma) = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_2 i_{\sigma(2)}} \cdots E_{i_m i_{\sigma(m)}} E_{i_1 i_{\sigma(1)}}$.

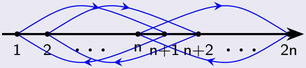
Digraph of a permutation

Definition (Digraph of a permutation)

The digraph $G(\sigma)$ of a permutation $\sigma \in S_m$ consists of these m vertices and m directed edges drawn as follows:

- the m permuted elements placed on a real line;
- directed edge from i to j iff $\sigma(i) = j$.

for example:

$G((1\ n+1)(2\ n+2)\cdots(n\ 2n)) =$ 

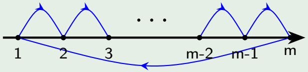
Casimir elements as $\mathfrak{gl}(N)$ weight system of permutations

Example

The Casimir element C_m in $ZU(\mathfrak{gl}(N))$

$$C_m = \sum_{i_1,\cdots,i_m=1}^{N} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_{m-1} i_m} E_{i_m i_1}$$

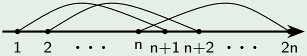
corresponds to the cyclic permutation
$1 \mapsto 2 \mapsto \cdots \mapsto m \mapsto 1 \in S_m$.
The digraph of the Casimir element C_m is the following one:

$G((1\ 2\ 3 \cdots m-1\ m)) =$ 

Chord diagram as a permutation

A chord diagram with n chords can be considered as an involution without fixed points on a set of $m = 2n$ elements. The value of $w_{\mathfrak{gl}(N)}$ on the corresponding involution is equal to the value of the $\mathfrak{gl}(N)$ weight system on the corresponding chord diagram.

Example

For the chord diagram $K_n =$ 

we have

$$w_{\mathfrak{gl}(N)}(K_n) = \sum_{i_1,\cdots,i_{2n}=1}^{N} E_{i_1 i_{n+1}} \cdots E_{i_n i_{2n}} E_{i_{n+1} i_1} E_{i_{n+2} i_2} \cdots E_{i_{2n} i_n}$$
$$= w_{\mathfrak{gl}(N)}((1\ n+1)(2\ n+2)\cdots(n\ 2n))$$

Theorem

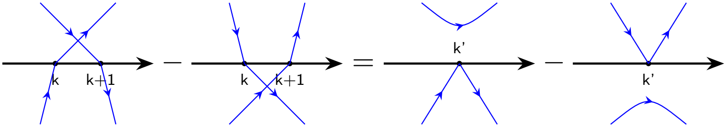
Theorem (Z.Yang, 2022)

The value of the $w_{\mathfrak{gl}_N}$ invariant of permutations possesses the following properties:

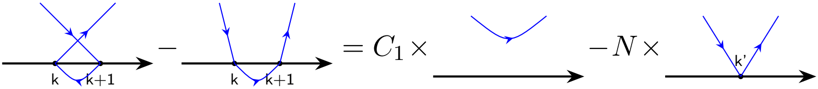
- *for an empty graph (with no vertices) the value of $w_{\mathfrak{gl}_N}$ is equal to 1, $w_{\mathfrak{gl}_N}(\bigcirc) = 1$;*
- *$w_{\mathfrak{gl}_N}$ is multiplicative with respect to concatenation of permutations;*
- *for a stanard cyclic permutation $($with the cyclic order on the set of permuted elements compatible with the permutation$)$, the value of $w_{\mathfrak{gl}_N}$ is the standard generator, $w_{\mathfrak{gl}_N}(1 \mapsto 2 \mapsto \cdots \mapsto k \mapsto 1) = C_k$.*

Recurrence Rule

For the graph of an arbitrary permutation σ in S_m, and for any two neighboring elements $k, k+1$ of the permuted set $\{1, 2, \ldots, m\}$, the value of the $w_{\mathfrak{gl}(N)}$ weight system satisfies:



In particular, for the special case $\sigma(k+1) = k$, the recurrence looks like follows:



Theorem

Theorem (Z.Yang, 2023)

The $\mathfrak{gl}(m|n)$-weight system for chord diagrams is a special case of the $\mathfrak{gl}(m|n)$-weight system for permutations, where we treat a chord diagram with k chords as an involution without fixed points on the set of $2k$ elements.

And $\mathfrak{gl}(m|n)$-weight system has the similar recurrence rule as $\mathfrak{gl}(n)$-weight system.

Theorem (Z.Yang, 2023)

The weight system $w_{\mathfrak{gl}(m|n)}$ for permutations is the result of substituting $m-n$ for C_0, and the k th Casimir element in $\mathfrak{gl}(m|n)$ for C_k, $k>0$, in the weight system $w_{\mathfrak{gl}}$.

Definition of $w_{\mathfrak{so}}$

Similarly to the case of the $w_{\mathfrak{gl}}$ weight system, we extend the definition of $w_{\mathfrak{so}}$ to the set of permutations (on any number of permuted elements).

$$w_{\mathfrak{so}(N)}(s) = \sum_{i_1,\cdots,i_m=1}^{N} X_{i_1 i_{s(1)}} X_{i_2 i_{s(2)}} \cdots X_{i_m i_{s(m)}} \in U\mathfrak{so}(N),$$

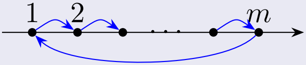
where X_{ij} are the standard generators of $\mathfrak{so}(N)$.
The invariant $w_{\mathfrak{so}}$ constructed below possesses an additional symmetry that does not hold for the $\mathfrak{gl}$ case:

> *assume that the permutation s' is obtained from s by the inversion of one of its independent cycles. In this case, the values $w_{\mathfrak{so}}(s)$ and $w_{\mathfrak{so}}(s')$ differ by the sign factor $(-1)^r$ where r is the length of the cycle.*

This symmetry leads to the following convention. Along with the digraphs of permutations, we consider more general graphs, which we call extended permutation graphs.

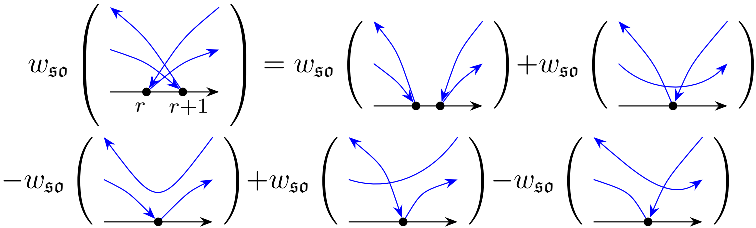
Definition (M. Kazarian, Z. Yang, in preparation)

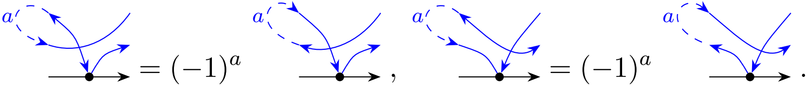
The universal polynomial invariant $w_{\mathfrak{so}}$ is the function on the set of permutations of any number of elements taking values in the ring of polynomials in the generators $C_0, C_2, C_4, \dots$ and defined by the following set of relations.

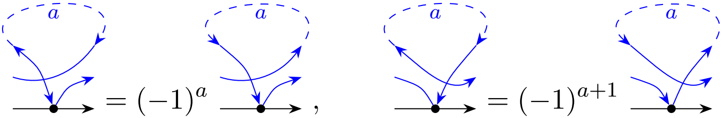
- $w_{\mathfrak{so}}$ is multiplicative with respect to the concatenation of the permutation graphs. As a corollary, for the empty graph (with no vertices) the value of $w_{\mathfrak{so}}$ is equal to 1;
- A change of orientation of any cycle of length r in the graph results in multiplication of the value of the invariant $w_{\mathfrak{so}}$ by $(-1)^r$.
- $w_{\mathfrak{so}}\Bigg($ 

$\Bigg) = C_m, \qquad m$ is even.

- **(Recurrence Rule)**







Theorem (M. Kazarian, Z. Yang, in preparation)

1. The defining relations determine the invariant $w_{\mathfrak{so}}$ *uniquely.*
2. For chord diagrams (corresponding to involutions without fixed points) the invariant $w_{\mathfrak{so}}$ *is a weight system, i.e. it satisfies the* 4*-term relation.*
3. The extension to the set of permutations for the weight systems associated with the Lie algebras $\mathfrak{so}(N)$ *and* $\mathfrak{sp}(2M)$*, and Lie superalgebra* $\mathfrak{osp}(N|2M)$ *take values in the center of the corresponding universal enveloping algebras and obey the defining relations of the universal* $w_{\mathfrak{so}}$ *weight system, and with* $C_0 = N$ *for the Lie algebra* $\mathfrak{so}(N)$*,* $C_0 = -2M$ *for the Lie algebra* $\mathfrak{sp}(2M)$*, and* $C_0 = N - 2M$ *for the Lie superalgebra* $\mathfrak{osp}(N|2M)$*.*

$\mathfrak{p}$ and $\mathfrak{q}$ weight system

The construction can be also extend to $\mathfrak{p}(N)$ and $\mathfrak{q}(N)$ by some generalization. Lie superalgebra $\mathfrak{p}(N)$ has trivial center of the universal enveloping algebra $U(\mathfrak{p}(N))$, so the correspond the $\mathfrak{p}(N)$ weight system on permutations is also trivial. The $\mathfrak{q}(N)$ weight system on permutations is not trivial, but it takes value 0 on all chord diagrams, and even on all permutations having at least one cycle of even length.

Thank you for your attention!