Space of initial values and symmetries of four dimensional Painlevé equations

Tomoyuki Takenawa (竹縄　知之)

Tokyo University of Marine Science and Technology

BIMSA, 10th, July, 2024

Representation Theory, Integrable Systems and Related Topics
Satellite conference for ICBS-2024

Contents

1. Case of the a Noumi-Yamada's Painlvé system: interesting as a dynamical system
2. Case of the 4D Garnier system: interesting in algebraic structure.

Noumi-Yamada's $A_5^{(1)}$ equation [Carstea-Takenawa 2019]

Let us consider a birational map $\varphi : \mathbb{C}^4 \to \mathbb{C}^4; (q_1, q_2, p_1, p_2) \mapsto (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$:

$$\begin{cases} \bar{q}_1 &= -q_1 - p_1 + aq_2^{-1} + b_1 \\ \bar{p}_1 &= q_1 \\ \bar{q}_2 &= -q_2 - p_2 + aq_1^{-1} + b_2 \\ \bar{p}_2 &= q_2 \end{cases},$$

where a, b_1, b_2 are fixed parameters. Then φ has two conserved quantities:

$$\begin{aligned} I_1 =& (q_1 p_2 - q_2 p_1)^2 + b_1 b_2 (q_1 p_2 + q_2 p_1) \\ & + b_1 \big(a(p_2 + q_2) - q_1 p_2^2 - q_2^2 p_1\big) + b_2 \big(a(q_1 + p_1) - q_1^2 p_2 - q_2 p_1^2\big) \\ I_2 =& \big(a(q_1 + p_1) + q_1 p_1 (b_2 - q_2 - p_2)\big)\big(a(q_2 + p_2) + q_2 p_2 (b_1 - q_1 - p_1)\big). \end{aligned}$$

These conserved quantities were found by examining the space of initial conditions.

This system is turned out to be a Bäcklund transformation of the autonomous version of the $A_5^{(1)}$ member of Noumi-Yamada's $A_n^{(1)}$ higher order Painlevé system [Noumi-Yamada 1998].
The conserved quantities give the Hamiltonian flows as:

$$\frac{dq_1}{dt} = \frac{\partial I_i}{\partial p_1}, \quad \frac{dp_1}{dt} = -\frac{\partial I_i}{\partial q_1}$$
$$\frac{dq_2}{dt} = \frac{\partial I_i}{\partial p_2}, \quad \frac{dp_2}{dt} = -\frac{\partial I_i}{\partial q_2}$$

and commute with each other as:

$$\{I_1, I_2\} = \left(\frac{\partial I_1}{\partial q_1}\frac{\partial I_2}{\partial p_1} - \frac{\partial I_1}{\partial p_1}\frac{\partial I_2}{\partial q_1}\right) + \left(\frac{\partial I_1}{\partial q_2}\frac{\partial I_2}{\partial p_2} - \frac{\partial I_1}{\partial p_2}\frac{\partial I_2}{\partial q_2}\right) = 0.$$

Question 1

Can the discrete system be written in a Lax form:$\overline{L}M = ML$? Can we solve in terms of theta functions?
At least Noumi-Yamada's $A_n^{(1)}$ system (of ODEs) can be written [Aoki et al. 2002, Takei 2004].
Tha matrix L may be

$$L = \begin{bmatrix} a & q_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -p_2 & 1 & 0 & 0 \\ 0 & 0 & a & b_1 - q_1 - p_1 & 1 & 0 \\ 0 & 0 & 0 & a & -q_2 & 1 \\ h & 0 & 0 & 0 & 0 & p_1 \\ h(-b_2 + q_2 + p_2) & h & 0 & 0 & 0 & a \end{bmatrix},$$

since its eigenpolynomial

$$|L - xI| = h^2 + x^2(x-a)^4 + h(-2x^3 + (4a + b_1 b_2)x^2 - (I_1 + 2a^2)x + I_2)$$

gives conserved quantities, but the modification matrix M is unknown. (We can find M numerically, but its expression is huge.)

Question 2: Relation to the Quispel-Roberts-Thompson map

For 3×3 matrices $\mathbf{A}$ and $\mathbf{B}$, consider a pencil of biquadratic curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$P(x, y; K) = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{y} + K\mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{y} = 0, \tag{1}$$

where $\mathbf{x} = (x^2, x, 1)^T$, $\mathbf{y} = (y^2, y, 1)^T$, and define a horizontal switch $r_x : (x, y) \to (\bar{x}, y)$ and a vertical switch $r_y : (x, y) \to (x, \bar{y})$ on the biquadratic curve. The composition of the two involutions $\varphi = r_y \circ r_x$ is called the QRT map [Quispel-Roberts-Thompson 1989] (decomposition is found in [Iatrou-Roberts 2002, Tsuda 2004])

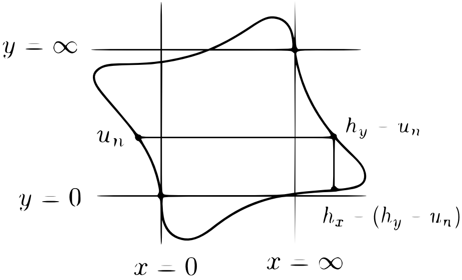


Figure: [Li-Takenawa 2022]

The 4D map φ can be decomposed to involutions

$$r_p : \begin{cases} \bar{q}_1 &=& q_1 \\ \bar{p}_1 &=& -q_1 - p_1 + aq_2^{-1} + b_1 \\ \bar{q}_2 &=& q_2 \\ \bar{p}_2 &=& -q_2 - p_2 + aq_1^{-1} + b_2 \end{cases} \quad \text{and } \sigma_{qp} : \begin{cases} \bar{q}_1 &=& p_1 \\ \bar{p}_1 &=& q_1 \\ \bar{q}_2 &=& p_2 \\ \bar{p}_2 &=& q_2 \end{cases},$$

which conserve I_1 and I_2. Therefore, it would be nice to be able to construct the reflection map from conserved quantities, but there are eight solutions for

$$I_i(q_1, \bar{p}_1, q_2, \bar{p}_2) = I_i(q_1, p_1, q_2, p_2), \quad (i = 1, 2).$$

Singularity confinement after Grammaticos-Ramani

Let us consider a birational map

$$\varphi : \begin{cases} \bar{q}_1 &= -q_1 - p_1 + a q_2^{-1} + b_1 \\ \bar{p}_1 &= q_1 \\ \bar{q}_2 &= -q_2 - p_2 + a q_1^{-1} + b_2 \\ \bar{p}_2 &= q_2 \end{cases},$$

Let us compactify the phase space to $(\mathbb{P}^1)^4$, where $\mathbb{P}^1$ is the Riemann sphere. Then, we find the following singularity sequence for φ: (ε is a small parameter, $|\varepsilon| \ll 1$)

$$\begin{aligned} (\varepsilon, p_1, q_2, p_2)\text{: 3 dim} \to & (-p_1 + a q_2^{-1} + b_1, \varepsilon, a\varepsilon^{-1}, q_2)\text{: 2 dim} \\ \to & (p_1 - a q_2^{-1}, -p_1 + a q_2^{-1} + b_1, -a\varepsilon^{-1}, a\varepsilon^{-1})\text{: 1 dim} \\ \to & (-\varepsilon, p_1 - a q_2^{-1}, q_2', -a\varepsilon^{-1})\text{: 2 dim} \\ \to & (q_1'', -\varepsilon, q_2'', p_2'')\text{: 3 dim}, \end{aligned}$$

where only the principal terms of the Laurent series are written.

Another singularity sequence is

$$\begin{aligned}(q_1, p_1, \varepsilon^{-1}, p_2)\text{: 3 dim} \to & (-p_1 - q_1 + b_1, q_1, -\varepsilon^{-1}, \varepsilon^{-1})\text{: 2 dim}\\ \to & (p_1, -p_1 - q_1 + b_1, q_2', -\varepsilon^{-1})\text{: 3 dim}\\ \to & (q_1'', p_1'', \varepsilon^{-1}, p_2'')\text{: Returned}\end{aligned}$$

In these two sequences, some 3 dimensional subvarieties are contracted to lower dimensional ones (called "singularity"), along which we want to blow-up. Since in order to eliminate these singularities we need infinitely near blow-ups, we should also consider the following singularity sequences which become bases of the above blow-ups.

$$\begin{aligned}(q_1, p_1, c_1\varepsilon^{-1}, c_2\varepsilon^{-1})\text{: 2 dim} \to & (q_1', p_1', c_1'\varepsilon^{-1}, c_2'\varepsilon^{-1})\text{: Returned}\\ (q_1, c_1\varepsilon, c_2\varepsilon^{-1}, p_2)\text{: 2 dim} \to & (-q_1 + b_1, q_1, -c_2\varepsilon^{-1}, c_2\varepsilon^{-1})\text{: 1 dim}\\ \to & (c_1'\varepsilon, -q_1 + b_1, p_1, -c_2\varepsilon^{-1})\text{: 2 dim}\\ \to & (q_1', c_1'\varepsilon, c_2'\varepsilon^{-1}, p_2')\text{: Returned}\end{aligned}$$

Space of initial conditions

Since the system is symmetric for the exchange of (q_1, p_1) and (q_2, p_2), there is a counterpart of the above sequences. Let X be a rational variety obtained by successive 16 blow-ups from $(\mathbb{P}^1)^4$ along the singularities. In this case, all the center manifolds of blow-ups are two-dimensional.

Theorem 1

φ can be lifted to a pseudo-automorphism on X (automorphism except finite number of subvarieties of co-dimension higher than two).

Neron-Séveri bilattice

Pick up a singular effective anti-canonical divisor. In our case it is a sum of 14 prime divisors $-K_X = F_1 + F_2 + \cdots + F_{14}$. Let $X_{\mathbf{a}} = X_{a_1, a_2, \cdots, a_5; b_1, b_2}$ be a rational variety obtained similar to X so that the decomposition F still holds.

Let H_i be the linear equivalent class of a divisor $x_i = \mathrm{constant}$ (under relabeling as $H_1 = H_{q_1}$, $H_2 = H_{q_2}$, $H_3 = H_{p_1}$, $H_4 = H_{p_2}$) and E_i is the exceptional divisor class of the i-th blow up, while $h_i's$ and e_i's are a basis of the dual space satisfying

$$\langle H_i, h_j \rangle = \delta_{ij}, \quad \langle H_i, e_j \rangle = \langle E_i, h_j \rangle = 0, \quad \langle E_i, e_j \rangle = -\delta_{ij}.$$

Then,

$$H^2(X_{\mathbf{a}}, \mathbb{Z}) = \bigoplus_{i=1}^{4} \mathbb{Z} H_i \oplus \bigoplus_{i=1}^{16} \mathbb{Z} E_i$$

$$H_2(X_{\mathbf{a}}, \mathbb{Z}) = \bigoplus_{i=1}^{4} \mathbb{Z} h_i \oplus \bigoplus_{i=1}^{16} \mathbb{Z} e_i$$

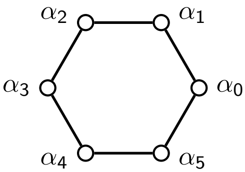
give the Neron-Séveri bilattice.

Root system

Let us define root vectors α_i ($i = 0, 1, \ldots, 5$) and co-root vectors as

$$
\begin{aligned}
&\alpha_0 = H_{q_1} + H_{p_1} - E_3 - E_4 - E_9 - E_{10}, \quad \alpha_1 = H_{q_2} - E_{15} - E_{16} \\
&\alpha_2 = H_{p_1} - E_5 - E_6, \quad \alpha_3 = H_{q_2} + H_{p_2} - E_1 - E_2 - E_{11} - E_{12} \\
&\alpha_4 = H_{q_1} - E_7 - E_8, \quad \alpha_5 = H_{p_2} - E_{13} - E_{14} \\
&\check{\alpha}_0 = h_{q_2} + h_{p_2} - e_1 - e_2 - e_3 - e_4, \quad \check{\alpha}_1 = h_{p_1} - e_{15} - e_{16} \\
&\check{\alpha}_2 = h_{q_2} - e_5 - e_6, \quad \check{\alpha}_3 = h_{q_1} + h_{p_1} - e_9 - e_{10} - e_{11} - e_{12} \\
&\check{\alpha}_4 = h_{p_2} - e_7 - e_8, \quad \check{\alpha}_5 = h_{q_1} - e_{13} - e_{14}.
\end{aligned}
$$

Then, the pairing $\langle \alpha_i, \check{\alpha}_j \rangle$ defined by the intersection form induces the affine root system of type $A_5^{(1)}$.



Theorem 2

The mapping w_{α_i} defined by

$$w_{\alpha_i}(D) := D - 2\frac{\langle D, \check{\alpha}_i\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\alpha_i, \quad w_{\alpha_i}(d) := d - 2\frac{\langle \alpha_i, d\rangle}{\langle \alpha_i, \check{\alpha}_i\rangle}\check{\alpha}_i$$

for $D \in H^2(X_{\mathbf{a}}, \mathbb{Z})$ and $d \in H_2(X_{\mathbf{a}}, \mathbb{Z})$ can be realized as a birational map from $X_{\mathbf{a}}$ to $X_{w(\mathbf{a})}$ (coincides with the Bäcklund transformation given by Noumi-Yamada).

The push-forward action of φ on the root lattice is

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_4 + \alpha_5, -\alpha_5, \alpha_0 + \alpha_5, \alpha_1 + \alpha_2, -\alpha_2, \alpha_2 + \alpha_3)$$

and hence that of φ^4 is a translation

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) + \delta(0, -1, 1, 0, -1, 1),$$

where $\delta = -K_X = \sum_{i=0}^{5} \alpha_i$.

4D Garnier system [Takenawa 2024]

$$\frac{dq_i}{ds_j} = \frac{\partial H_j}{\partial p_i}, \quad \frac{dp_i}{ds_j} = -\frac{\partial H_j}{\partial q_i} \quad (i, j = 1, 2)$$

with

$$\begin{aligned}
s_1(s_1-1)H_1 =& \left(q_1(q_1-1)(q_1-s_1) - \frac{s_1(s_1-1)}{s_1-s_2}q_1q_2\right)p_1^2 \\
& + 2q_1q_2\left(q_1 + \frac{s_1(s_2-1)}{s_1-s_2}\right)p_1p_2 + q_1q_2\left(q_2 - \frac{s_2(s_1-1)}{s_1-s_2}\right)p_2^2 \\
& - \Big\{(\kappa_0 - d)q_1(q_1-1) + \kappa_1 q_1(q_1-s_1) + \theta_1(q_1-1)(q_1-s_1) \\
& \quad + \theta_2 q_1\left(q_1 + \frac{s_1(s_2-1)}{s_1-s_2}\right) - \theta_1\frac{s_1(s_1-1)}{s_1-s_2}q_2\Big\}p_1 \\
& + \left((2_0+\kappa_\infty)q_1q_2 + \theta_2 q_1\frac{s_2(s_1-1)}{s_1-s_2} - \theta_1 q_2\frac{s_1(s_2-1)}{s_1-s_2}\right)p_2 \\
& + a_0(a_0+\kappa_\infty)q_1 \\
s_2(s_2-1)H_2 =& \{\text{replacing as } q_1 \leftrightarrow q_2,\ p_1 \leftrightarrow p_2,\ s_1 \leftrightarrow s_2,\ \theta_1 \leftrightarrow \theta_2 \text{ in } H_1\},
\end{aligned}$$

and $d = 2\alpha_0 + \kappa_0 + \kappa_1 + \kappa_\infty + \theta_1 + \theta_2$

Known symmetries

[Kimura 1990] except for w_{α_0}] found in [Tsuda 2003]

Actions on parameters:

	$\bar{\kappa}_0$	$\bar{\kappa}_1$	$\bar{\kappa}_\infty$	$\bar{\theta}_1$	$\bar{\theta}_2$	$\bar{\alpha}_0$	$\bar{s}_1$	$\bar{s}_2$
w_{κ_0}	$-\kappa_0$	κ_1	κ_∞	θ_1	θ_2	$\alpha_0+\kappa_0$	s_1	s_2
w_{κ_1}	κ_0	$-\kappa_1$	κ_∞	θ_1	θ_2	$\alpha_0+\kappa_1$	s_1	s_2
w_{κ_∞}	κ_0	κ_1	$-\kappa_\infty$	θ_1	θ_2	$\alpha_0+\kappa_\infty$	s_1	s_2
w_{θ_1}	κ_0	κ_1	κ_∞	$-\theta_1$	θ_2	$\alpha_0+\theta_1$	s_1	s_2
w_{θ_2}	κ_0	κ_1	κ_∞	θ_1	$-\theta_2$	$\alpha_0+\theta_2$	s_1	s_2
w_{α_0}	$d-\kappa_0$	$d-\kappa_1$	$-\kappa_\infty$	$-\theta_1$	$-\theta_2$	$-\alpha_0$	s_1	s_2
σ_1	κ_1	κ_0	κ_∞	θ_1	θ_2	α_0	s_1^{-1}	s_2^{-1}
σ_2	κ_0	κ_∞	κ_1	θ_1	θ_2	α_0	$\frac{s_1}{s_1-1}$	$\frac{s_2}{s_2-1}$
σ_3	κ_0	κ_1	θ_1	κ_∞	θ_2	α_0	s_1^{-1}	$s_1^{-1}s_2$
σ_4	κ_0	κ_1	κ_∞	θ_2	θ_1	α_0	s_2	s_1

Actions on dependent and independent variables:

	$\bar{q}_1$	$\bar{q}_2$	$\bar{r}_1$	$\bar{r}_2$
w_{κ_0}	q_1	q_2	$r_1 - \frac{\kappa_0 q_1}{s_1 Q_{12}^s}$	$r_2 - \frac{\kappa_0 q_2}{s_2 Q_{12}^s}$
w_{κ_1}	q_1	q_2	$r_1 - \frac{\kappa_1 q_1}{Q_{12}}$	$r_2 - \frac{\kappa_1 q_2}{Q_{12}}$
w_{κ_∞}	q_1	q_2	r_1	r_2
w_{θ_1}	q_1	q_2	$r_1 - \theta_1$	r_2
w_{θ_2}	q_1	q_2	r_1	$r_2 - \theta_2$
w_{α_0}	$\frac{s_1 r_1 (r_1 - \theta_1)}{q_1 R_{12}(R_{12} + \kappa_\infty)}$	$\frac{s_2 r_2 (r_2 - \theta_2)}{q_2 R_{12}(R_{12} + \kappa_\infty)}$	$-r_1$	$-r_2$
σ_1	$s_1^{-1} q_1$	$s_2^{-1} q_2$	r_1	r_2
σ_2	$\frac{q_1}{Q_{12}}$	$\frac{q_2}{Q_{12}}$	$r_1 - q_1 R_{12}$	$r_2 - q_2 R_{12}$
σ_3	q_1^{-1}	$-q_1^{-1} q_2$	$-R_{12}$	r_2
σ_4	q_2	q_1	r_2	r_1

where

$$
\begin{aligned}
r_1 &= q_1 p_1, \quad r_2 = q_2 p_2 \\
Q_{12} &= q_1 + q_2 - 1 \\
Q_{12}^s &= q_1/s_1 + q_2/s_2 - 1 \\
R_{12} &= r_1 + r_2 + \alpha_0.
\end{aligned}
$$

Theorem 3

(i) *Every generator except w_{α_0} is lifted to a pseudo-isomorphism between rational projective varieties obtained by successive 10 blow-ups from $\mathbb{P}^2 \times \mathbb{P}^2$ such that the center of each blow-up, C_i ($i = 1, \dots, 10$) is two-dimensional for $i \neq 7, 9$ and one-dimensional for $i = 7, 9$.*

(ii) *Every generator including w_{α_0} is lifted to a pseudo-isomorphism between rational projective varieties obtained by successive $10 + 11 = 21$ blow-ups from $\mathbb{P}^2 \times \mathbb{P}^2$ such that the center of each blow-up, where C_i ($i = 11, \dots, 21$) is two-dimensional for $i \neq 11, 12, 13$, zero-dimensional for $i = 14, \dots, 19$ and one-dimensional for $i = 20, 21$.*

Remark 1

The space of Claim (i) is similar to but simpler than Kimura's SIC for the DE [Kimura 93]. The blowup in Claim (ii) only appear when discrete symmetry is considered, since their base locus are included in "vertical leafs" through which no solution passes.

Action on the Picard group with 21 blow-ups

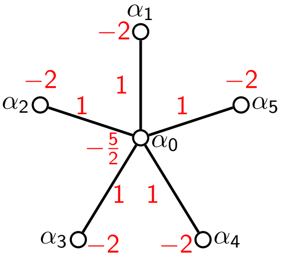
w_{κ_0}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{9,10,15,17,19,20}$ $\mathcal{E}_9 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{10,15,17,19,20}, \quad \mathcal{E}_{10} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{9,15,17,19,20}$
w_{κ_1}	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{7,8,14,16,18,20}$ $\mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{8,14,16,18,20}, \quad \mathcal{E}_8 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{7,14,16,18,20}$
w_{κ_∞}	$\mathcal{E}_5 \leftrightarrow \mathcal{E}_6$
w_{θ_1}	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_2$
w_{θ_2}	$\mathcal{E}_3 \leftrightarrow \mathcal{E}_4$
w_{α_0}	$\mathcal{H}_q \leftrightarrow 2\,\mathcal{H}_q + 2\,\mathcal{H}_r - \mathcal{E}_{1,2,3,4,5,6,11,12,13,14,15,16,17,18,19}$ $\mathcal{E}_1 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{1,14,15}, \quad \mathcal{E}_2 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{2,14,15}, \quad \mathcal{E}_3 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{3,16,17}$ $\mathcal{E}_4 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{4,16,17}, \quad \mathcal{E}_5 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{5,18,19}, \quad \mathcal{E}_6 \leftrightarrow \mathcal{H}_r - \mathcal{E}_{6,18,19}$ $\mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10}$ $\mathcal{E}_{11} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{1,2,12,13,14,15}, \quad \mathcal{E}_{12} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{3,4,11,13,16,17}$ $\mathcal{E}_{13} \leftrightarrow \mathcal{H}_q - \mathcal{E}_{5,6,11,12,18,19} \quad \mathcal{E}_{20} \leftrightarrow \mathcal{E}_{21}$
σ_1	$\mathcal{E}_7 \leftrightarrow \mathcal{E}_9, \quad \mathcal{E}_8 \leftrightarrow \mathcal{E}_{10}$ $\mathcal{E}_{14} \leftrightarrow \mathcal{E}_{15}, \quad \mathcal{E}_{16} \leftrightarrow \mathcal{E}_{17}, \quad \mathcal{E}_{18} \leftrightarrow \mathcal{E}_{19}$
σ_2	$\mathcal{H}_r \leftrightarrow \mathcal{H}_q + \mathcal{H}_r - \mathcal{E}_{5,7,14,16,18,19}, \quad \mathcal{E}_5 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{7,14,16,18,20}$ $\mathcal{E}_6 \leftrightarrow \mathcal{E}_8, \quad \mathcal{E}_7 \leftrightarrow \mathcal{H}_q - \mathcal{E}_{5,11,12,18,19} \quad \mathcal{E}_{11} \leftrightarrow \mathcal{E}_{16}, \quad \mathcal{E}_{12} \leftrightarrow \mathcal{E}_{14}, \quad \mathcal{E}_{19} \leftrightarrow \mathcal{E}_{20}$
σ_3	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_5, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_6$ $\mathcal{E}_{11} \leftrightarrow \mathcal{E}_{13}, \quad \mathcal{E}_{14} \leftrightarrow \mathcal{E}_{18}, \quad \mathcal{E}_{15} \leftrightarrow \mathcal{E}_{19}$
σ_4	$\mathcal{E}_1 \leftrightarrow \mathcal{E}_3, \quad \mathcal{E}_2 \leftrightarrow \mathcal{E}_4$ $\mathcal{E}_{11} \leftrightarrow \mathcal{E}_{12}, \quad \mathcal{E}_{14} \leftrightarrow \mathcal{E}_{16}, \quad \mathcal{E}_{15} \leftrightarrow \mathcal{E}_{17}$

Root system for the 4D Garnier

Let $X_{\mathbf{a}}$ be the space of initial conditions obtained by the first 10 blow-ups. Define root vectors α_i ($i = 0, 1, \ldots, 5$) and co-root vectors as

$$\begin{aligned}
&\alpha_0 = \tfrac{1}{2}(H_1 + 2H_2 - 2E_1 - 2E_3 - 2E_5), \quad \alpha_1 = H_1 - E_9 - E_{10} \\
&\alpha_2 = H_1 - E_7 - E_8, \quad \alpha_3 = E_5 - E_6 \\
&\alpha_4 = E_1 - E_2, \quad \alpha_5 = E_3 - E_4
\end{aligned}$$

$$\begin{aligned}
&\check{\alpha}_0 = h_1 - e_1 - e_3 - e_5, \quad \check{\alpha}_1 = h_2 - e_9 - e_{10} \\
&\check{\alpha}_2 = h_2 - e_7 - e_8, \quad \check{\alpha}_3 = e_5 - e_6 \\
&\check{\alpha}_4 = e_1 - e_2, \quad \check{\alpha}_5 = e_3 - e_4.
\end{aligned}$$



.

Then, w_{α_i} $(i \neq 0)$ coincides with one of original w_i's and acts on the Néron-Severi bi-lattice as

$$w_{\alpha_i}(D) = D - 2\frac{\langle D, \check{\alpha}_i \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\alpha_i, \quad w_{\alpha_i}(d) = d - 2\frac{\langle \alpha_i, d \rangle}{\langle \alpha_i, \check{\alpha}_i \rangle}\check{\alpha}_i$$

for $D \in H^2(X_{\mathbf{a}}, \mathbb{Z})$ and $d \in H_2(X_{\mathbf{a}}, \mathbb{Z})$. If we apply the above formula to w_{α_0} with $D = H_1$, we obtain

$$H_1 \mapsto H_1 + \frac{2}{5}(H_1 + 2H_2 - E_1 - E_3 - E_5),$$

which is not in $H^2(X_{\mathbf{a}}, \mathbb{Z})$ and hence w_{α_0} is impossible to be realized as a birational map.

However, surprisingly the following theorem holds and gives the reason why we take α_0 as above.

Theorem 4

Kac's translation (§6.5 in [Kac 1990]):

$$T_{\alpha_i}(D) = D + \langle D, \check{\delta} \rangle \alpha_i + \langle D, \check{\delta} - \check{\alpha}_i \rangle \delta$$

with

$$\delta = 2\alpha_0 + \sum_{i=1}^{5} \alpha_i = 3\,\mathcal{H}_q + 2\,\mathcal{H}_r - \mathcal{E}_{1,2,3,4,5,6,7,8,9,10} \tag{2}$$

$$\check{\delta} = 2\check{\alpha}_0 + \sum_{i=1}^{5} \check{\alpha}_i = 2h_q + 2h_r - e_{1,2,3,4,5,6,7,8,9,10} \tag{3}$$

can be realized as a pseudo-isomorphism for $i = 1, 2, 3, 4, 5$. Moreover, $T^2_{\alpha_0}$ can be realized as a pseudo-isomorphism.

Especially, T_{α_1} acts on $(\alpha_0, \alpha_1, \ldots, \alpha_5)$ as

$$T_{\alpha_1} : (\alpha_0, \alpha_1, \ldots, \alpha_5) \mapsto (\alpha_0, \alpha_1, \ldots, \alpha_5) + \delta(-1, 2, 0, 0, 0, 0)$$

and $T^2_{\alpha_0}$ acts on $(\alpha_0, \alpha_1, \ldots, \alpha_5)$ as

$$T^2_{\alpha_0} : (\alpha_0, \alpha_1, \ldots, \alpha_5) \mapsto (\alpha_0, \alpha_1, \ldots, \alpha_5) + \delta(5, -2, -2, -2, -2, -2).$$

Proof.

It tuns out that the action of

$$T_1 = (\sigma_1 w_{\theta_2} w_{\theta_1} w_{\kappa_2} w_{\kappa_1} w_{\alpha_0})^2$$

on the space with 21 blow-ups is trivial on the sub-lattice expanded by $\mathcal{E}_{11}, \ldots, \mathcal{E}_{21}$, and hence T_1 is a pseudo-isomorphism on the space with the first 10 blow-ups.
Similarly T_{α_i} is realized.
Using these translations, we can realize T_{α_0} as

$$T_{\alpha_0}^2 = T_1 T_2 T_3 T_4 T_5.$$

$\square$

Concluding remark

In this talk we constructed the space of initial conditions for some 4D Painlevé systems.

In higher dimensional and autonomous cases, there are problems of constructing solutions using Lax forms and constructing dynamical systems from conserved quantities.

It would be interesting to understand more about root systems whose self inner product is a fractional number.