Ruijsenaars duality for B, C, D Toda chains

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GENERALIZED TODA CHAINS

- $H(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{2(q_i - q_{i+1})}$ — A-type Toda chain

- other root systems

  simple roots

  $$H(p,q) = \frac{1}{2}(p,p) + \sum_{\alpha\in\Pi} c_\alpha e^{2\alpha(q)} \qquad (p,q)\in\mathfrak{h},\quad \alpha\in\Pi\subset\mathfrak{h}^*$$

- classical integrable system

  n independent Poisson-commuting functions $\{H_i, H_j\} = 0$

  $$H_1(p,q) = \sum_i p_i$$

  $$H_2(p,q) = H(p,q)$$

  $$\vdots$$

  $$H_\ell(p,q) = \frac{1}{\ell}\sum_i p_i^\ell + (\text{lower terms in } p)$$

RUIJSENAARS DUALITY

- classical integrable system: $\{H_i, H_j\} = 0 \quad i,j = 1,\dots,n$
- another Poisson-commuting set: $\{q_i, q_j\} = 0 \quad i,j = 1,\dots,n$

Ruijsenaars duality:
two integrable systems on the same phase space

$$H_i(p,q) \longleftrightarrow \hat{q}_i$$
$$q_i \longleftrightarrow \hat{H}_i(\hat{p},\hat{q})$$

action variables $\updownarrow$ coordinates

Dualities square for Calogero–Ruijsenaars integrable systems

coords. \ mom.	rat.	trig.	ell.
rat.	rat. CM	rat. RS	dual to ell. CM
trig.	trig. CM	trig. RS	dual to ell. RS
ell.	ell. CM	ell. RS	double elliptic

GOLDFISH MODELS

- Ruijsenaars dual to Toda chains

$$\hat{H}(\hat{p},\hat{q}) = \sum_{i=1}^{n} e^{2\hat{p}_i} \prod_{j\neq i} \frac{1}{\hat{q}_i - \hat{q}_j}$$

momentum in exp – trigonometric
coordinates – rational dependence
(opposite to the Toda case)

trigonometric CM system

$$H(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i\neq j}^{n} \frac{g^2}{2\sin^2(q_i - q_j)}$$

Ruijsenaars duality

rational RS system

$$\hat{H}(\hat{p},\hat{q}) = \sum_{i=1}^{n} e^{2\hat{p}_i} \prod_{j\neq i} \frac{\hat{q}_i - \hat{q}_j + \eta}{\hat{q}_i - \hat{q}_j}$$

Inozemtsev limit

strong coupling limit

$$H(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{2(q_i - q_{i+1})}$$

$$\hat{H}(\hat{p},\hat{q}) = \sum_{i=1}^{n} e^{2\hat{p}_i} \prod_{j\neq i} \frac{1}{\hat{q}_i - \hat{q}_j}$$

(open) Toda chain

(rational) Goldfish model

HAMILTONIAN REDUCTION

- (M,ω) – symplectic manifold with Hamiltonian action of Lie group G
  Ad^*_G–equivariant momentum map $\mu: M \to \mathfrak{g}^*$ $\langle \mu(x), \xi \rangle = H_\xi(x)$
- reduced phase space $M_\lambda = \mu^{-1}(\lambda)/\mathrm{Stab}^{\mathrm{Ad}^*}(\lambda)$ for fixed $\lambda \in \mathfrak{g}^*$
- symplectic form ω_λ:

$$\begin{array}{ccc} \mu^{-1}(\lambda) & \xhookrightarrow{i} & M \\ {\scriptstyle \pi}\downarrow & & \\ M_\lambda & & \end{array} \qquad i^*\omega = \pi^*\omega_\lambda$$

- source of integrable systems:
  simple functions $f, g \in C(M)^G$ $\longrightarrow$ nontrivial functions $\tilde{f}, \tilde{g} \in C(M_\lambda)$
  large phase space $\{f, g\}_M = 0$ on reduced space $\{\tilde{f}, \tilde{g}\}_{M_\lambda} = 0$

LIE GROUPS NOTATIONS

- G – matrix complex Lie group

$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$

$\theta(X) = -X^\dagger;\quad \mathfrak{k} = \mathfrak{g}|_{\theta=1}$

- $N_\pm$ – unipotent Lie groups $\quad \mathrm{Lie}(N_\pm) = \mathfrak{n}_\pm$

  K – maximal compact subgroup $\quad \mathrm{Lie}(K) = \mathfrak{k}$

skew-symmetric form

$$\Omega = \begin{pmatrix} 0 & P \\ -P & 0 \end{pmatrix}_{2n\times 2n}$$

$$P = \begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}_{n\times n}$$

- Type A

$$G = GL(n,\mathbb{C})$$

$$N_+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

- Type C

$$G = Sp(2n,\mathbb{C}) = \{ g \in GL(2n,\mathbb{C}) \mid g\Omega g^T = \Omega \}$$

upper-triangular

1's on the diagonal

$$N_+ = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & P(A^T)^{-1}P \end{array} \right) \middle| \ A = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$$

$$K = U(n)$$

both unitary

$$K = U(2n) \cap Sp(2n,\mathbb{C})$$

COTANGENT BUNDLE

- left translations trivialization $T^*G \simeq G \times \mathfrak{g}^* \ni (g, X)$

- canonical symplectic structure $\alpha = \mathrm{Tr}(X g^{-1} dg) \longleftarrow \{X_a, X_b\} = -f_{ab}^{c} X_c \leftarrow$ Lie-algebraic brackets

  $\omega = d\alpha \qquad \{g, g\} = 0 \leftarrow g$ matrix elements commute

- left and right Hamiltonian actions

  $L_h : (g, X) \longmapsto (hg, X) \qquad R_h : (g, X) \longmapsto (gh^{-1}, hXh^{-1})$

  $\mu_L(g, X) = gXg^{-1} \qquad \mu_R(g, X) = -X$

- subgroups actions

  N_+ left action $\qquad K$ right action

  $\mu_L(g, X) = \mathrm{Pr}_{\mathfrak{n}_+^*}(gXg^{-1}) \in \mathfrak{n}_+^* \simeq \mathfrak{n}_- \qquad \mu_R(g, X) = -\mathrm{Pr}_{\mathfrak{k}^*}(X) \in \mathfrak{k}^* \simeq \mathfrak{k}$

  use Tr to identify

REDUCED PHASE SPACE

- momentum maps

$$\mu_L(g,X) = \mathrm{Pr}_{\mathfrak{n}_-}(gXg^{-1}) = \lambda_L = \sum_{\alpha\in\Pi} e_{-\alpha}$$

$$\mu_R(g,X) = -\mathrm{Pr}_{\mathfrak{k}}(X) = \lambda_R = 0$$

$\alpha\in\Pi$: simple roots

maximal possible stabilizer — $N_+\times K$

$k\cdot 0\cdot k^{-1} = 0 \quad \forall k\in K$

$n\left(\sum_{\alpha\in\Pi} e_{-\alpha}\right)n^{-1} = \sum_{\alpha\in\Pi} e_{-\alpha} \quad \forall n\in N_+$

- reduced phase space

$$M_\lambda = \{(g,X)\mid \mu_L=\lambda_L,\ \mu_R=\lambda_R\}/N_+\times K$$

$\dim_{\mathbb{R}} M_L = 2\,\mathrm{rk}\,G$

- invariant commuting functions

$\mathrm{rk}\,G$ independent

$$I_\ell = \mathrm{Tr}\,X^\ell \qquad\qquad J_\ell = m_\ell(gg^+)$$

$\{\mathrm{Tr}\,X^\ell, \mathrm{Tr}\,X^m\} = 0$

as Casimirs $\{X_a, X_b\} = -f_{ab}^c X_c$

lowest-right $\ell\times\ell$ minors

$\{g,g\} = 0 \Rightarrow \{J_\ell, J_m\} = 0$

$$gg^+ \mapsto ng\underbrace{k^{-1}\cdot(k^{-1})^+}_{=1,\ k\text{-unitary}}g^+n^+ = n\cdot gg^+\cdot n^+ \Rightarrow \text{invariant}$$

n: upper-tr. with 1's diagonal

n^+: lower-tr. with 1's diagonal

TODA GAUGE

reduced phase space $\left\{ \mathrm{Pr}_{\mathfrak{k}}(X)=0,\ \mathrm{Pr}_{\mathfrak{n}_-}(gXg^{-1})=\sum_{\alpha\in\Pi} e_{-\alpha} \right\} / N_+\times K$

- Toda gauge — use $N_+\times K$ to diagonalize $g \mapsto ngk^{-1}$
  Iwasawa decomposition $G \to N_+\times A\times K$, A – diagonal matrices
  $g = \exp\left(\sum_{i=1}^{\mathrm{rk}G} q_i h_i\right) \quad q_i\in\mathbb{R}$
- Solve momentum equations
  $\mathrm{Pr}_{\mathfrak{k}}(X)=0 \Rightarrow X = p + \sum_{\alpha>0} X_\alpha (e_\alpha + e_{-\alpha})$, $\quad \mathrm{Pr}_{\mathfrak{n}_-}(gXg^{-1})=0 \Rightarrow \begin{cases} X_\alpha = e^{\alpha(q)}, & \alpha\in\Pi \\ X_\alpha = 0, & \alpha\notin\Pi \end{cases}$
- Toda chain Lax operator $X = p + \sum_{\alpha\in\Pi} e^{\alpha(q)}(e_\alpha + e_{-\alpha})$

$I_\ell = \mathrm{Tr}\, X^\ell$ – Toda chain Hamiltonians $\qquad J_\ell = m_\ell(gg^T)$ – simple functions of Toda coordinates

MOSER GAUGE

reduced phase space $\left\{ Pr_{\mathfrak{k}}(X)=0,\ Pr_{\mathfrak{n}_-}(gXg^{-1})=\sum_{\alpha\in\Pi} e_{-\alpha} \right\} / N_+ \times K$

- Moser gauge — use $X \mapsto kXk^{-1}$ to diagonalize $X=\sum \hat{q}_i h_i$ ← always possible $Pr_{\mathfrak{k}}(X)=0$
- N_+ gauge freedom — make g lower-triangular using Gauss decomposition $g\in B_-$ ← opposite Borel subgroup
- solve momentum equations

$$Pr_{\mathfrak{n}_-}(gXg^{-1})=\sum_{\alpha\in\Pi} e_{-\alpha} \Rightarrow gXg^{-1}-X=\sum_{\alpha\in\Pi} e_{-\alpha} \Rightarrow gX=Xg+\sum_{\alpha\in\Pi} e_{-\alpha}\cdot g$$

$g\in B_-$, $X\in\mathfrak{h}$ $\Rightarrow gXg^{-1}\in\mathfrak{b}_-=\mathfrak{h}\oplus\mathfrak{n}_-$, $Pr_{\mathfrak{h}}(gXg^{-1})=X$

- $I_\ell=\mathrm{Tr}\,X^\ell$ — simple functions of dual coordinates
  $J_\ell=m_\ell(gg^\dagger)$ — dual Hamiltonians

TYPE A: SOLVING MOMENTUM EQ

Moser gauge $X = \sum_{i=1}^{\mathrm{rk}G} \hat{q}_i h_i$ — diagonal matrix, real-valued $\hat{q}_i \in \mathbb{R}$

for $G = GL(n,\mathbb{C})$ $\quad X = \begin{pmatrix} \hat{q}_1 & & & 0 \\ & \hat{q}_2 & & \\ & & \ddots & \\ 0 & & & \hat{q}_n \end{pmatrix}$ $\quad \lambda_L = \sum_{\alpha\in\Pi} e_{-\alpha} = \begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$

- momentum equations: $gX - Xg = \lambda_L g \implies g_{ij}(\hat{q}_j - \hat{q}_i) = g_{i-1,j} \quad (i \neq j)$

no constraints on diagonal elements of g

recursive formula for g matrix elements starting from diagonal elements

- denote $g_{ii} = \hat{a}_i$, solution of momentum equations

$$g_{ij} = \hat{a}_j \cdot \prod_{k=j+1}^{i} \frac{1}{\hat{q}_j - \hat{q}_k} \quad (i > j)$$

lower-triangular

Type C: Solving Momentum Eq

Moser gauge $X = \sum_{i=1}^{\mathrm{rk}\,G} \hat{q}_i h_i$ — diagonal matrix, real-valued $\hat{q}_i \in \mathbb{R}$

for $G = Sp(2n, \mathbb{C})$ $\quad X = \left(\begin{array}{ccc|ccc} \hat{q}_1 & & & & & \\ & \ddots & & & 0 & \\ & & \hat{q}_n & & & \\ \hline & & & -\hat{q}_n & & \\ & 0 & & & \ddots & \\ & & & & & -\hat{q}_1 \end{array}\right)$ $\quad \lambda_L = \left(\begin{array}{cccc|cccc} 0 & & & & & & & \\ 1 & \ddots & & & & 0 & & \\ & \ddots & \ddots & & & & & \\ & & 1 & 0 & & & & \\ \hline & & & 1 & 0 & & & \\ & 0 & & & -1 & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & -1 & 0 \end{array}\right)$

- fix diagonal elements $\quad g = \begin{pmatrix} A & 0 \\ C & P(A^T)^{-1}P \end{pmatrix}$, $\quad A = \begin{pmatrix} \hat{a}_1 & & 0 \\ & \ddots & \\ * & & \hat{a}_n \end{pmatrix}$

- recursive relations from momentum equation
$$\begin{cases} (\hat{q}_j - \hat{q}_i) A_{ij} = A_{i-1,j} \\ (\hat{q}_{n+1-i} + \hat{q}_j) C_{ij} - \delta_{i,1} A_{nj} + C_{i-1,j} = 0 \end{cases}$$

- solution $\quad A_{ij} = \hat{a}_j \prod_{k=j+1}^{i} \frac{1}{\hat{q}_j - \hat{q}_k}, \quad C_{ij} = (-1)^{i-1} \hat{a}_j \prod_{k=j+1}^{n} \frac{1}{\hat{q}_j - \hat{q}_k} \prod_{m=1}^{i} \frac{1}{\hat{q}_{n+1-m} + \hat{q}_j}$

Type A: Dual Hamiltonians

- Hamiltonians using Cauchy-Binet formula

$$J_\ell = \sum_{|S|=\ell} \hat{a}_{s_i}^2 \times \prod_{r=1}^{\ell} \left(\prod_{\substack{j=s_r+1 \\ j\notin S}}^{n} \frac{1}{\hat{q}_{s_r}-\hat{q}_j} \right)^2$$

- Change of variables $e^{2\hat{p}_i} = \hat{a}_i^2 \cdot \prod_{j=i+1}^{n} (\hat{q}_i - \hat{q}_j) \cdot \prod_{j=1}^{i-1} \frac{1}{\hat{q}_i - \hat{q}_j}$

for more symmetric form of Hamiltonians

$$J_\ell = \sum_{|S|=\ell} \prod_{i=1}^{\ell} e^{2\hat{p}_{s_i}} \cdot \prod_{\substack{a\in S \\ b\notin S}} \frac{1}{\hat{q}_a - \hat{q}_b}$$

e.g. $J_1 = \sum_{i=1}^{n} e^{2\hat{p}_i} \prod_{j\neq i} \frac{1}{\hat{q}_i - \hat{q}_j}$

- strong coupling limit of rational Ruijsenaars-Schneider

$$H_\ell = \sum_{|S|=\ell} \prod_{i=1}^{\ell} e^{2\hat{p}_{s_i}} \cdot \prod_{\substack{a\in S \\ b\notin S}} \frac{\hat{q}_a - \hat{q}_b + \eta}{\hat{q}_a - \hat{q}_b}$$

Type C: Dual Hamiltonians

- $g = \begin{pmatrix} A & 0 \\ C & P(A^T)^{-1}P \end{pmatrix}$, $A = \begin{pmatrix} \hat{a}_1 & & 0 \\ & \ddots & \\ * & & \hat{a}_n \end{pmatrix}$

  only these matrix elements appear in $J_\ell = m_\ell(gg^+)$, $1 \le \ell \le n$

- $(P(A^T)^{-1}P)_{ij} = (-1)^{i-j}\, \hat{a}_{n+1-j}^{-1} \prod_{k=j+1}^{i} \frac{1}{\hat{q}_{n+1-k} - \hat{q}_{n+1-j}}$

- $C_{ij} = (-1)^{i-1}\, \hat{a}_j \prod_{k=j+1}^{n} \frac{1}{\hat{q}_j - \hat{q}_k} \prod_{m=1}^{i} \frac{1}{\hat{q}_{n+1-m} + \hat{q}_j}$

  these n lowest rows $=$ $GL(2n, \mathbb{C})$ matrix g with $\hat{a}_1, \dots, \hat{a}_n, \hat{a}_n^{-1}, \dots, \hat{a}_1^{-1}$ and $\hat{q}_1, \dots, \hat{q}_n, -\hat{q}_n, \dots, -\hat{q}_1$

- Cauchy-Binet + A type calculations $\longrightarrow$ Hamiltonians

- J_1 after change of variables $J_1 = \sum_{i=1}^{n} \left(e^{2\hat{p}_i} + e^{-2\hat{p}_i}\right) \frac{1}{2\hat{q}_i} \prod_{j \ne i} \frac{1}{(\hat{q}_i - \hat{q}_j)(\hat{q}_i + \hat{q}_j)}$

REVIEW OF ANSWERS

Toda system		Goldfish model
$H=\sum_{i=1}^{n}\frac{p_i^2}{2}+\sum_{i=1}^{n-1}e^{2(q_i-q_{i+1})}$	A	$\hat{H}=\sum_{i=1}^{n}e^{2\hat{p}_i}\prod_{j\neq i}\frac{1}{\hat{q}_i-\hat{q}_j}$
$H=\sum_{i=1}^{n}\frac{p_i^2}{2}+\sum_{i=1}^{n-1}e^{2(q_i-q_{i+1})}+e^{2q_n}$	B	$\hat{H}=\sum_{i=1}^{n}\left(e^{2\hat{p}_i}+e^{-2\hat{p}_i}\right)\frac{1}{2\hat{q}_i^2}\prod_{j\neq i}\frac{1}{(\hat{q}_i-\hat{q}_j)(\hat{q}_i+\hat{q}_j)}+\prod_{j=1}^{n}\frac{1}{\hat{q}_j^2}$
$H=\sum_{i=1}^{n}\frac{p_i^2}{2}+\sum_{i=1}^{n-1}e^{2(q_i-q_{i+1})}+\frac{1}{2}e^{4q_n}$	C	$\hat{H}=\sum_{i=1}^{n}\left(e^{2\hat{p}_i}+e^{-2\hat{p}_i}\right)\frac{1}{2\hat{q}_i}\prod_{j\neq i}\frac{1}{(\hat{q}_i-\hat{q}_j)(\hat{q}_i+\hat{q}_j)}$
$H=\sum_{i=1}^{n}\frac{p_i^2}{2}+\sum_{i=1}^{n-1}e^{2(q_i-q_{i+1})}+e^{2(q_{n-1}+q_n)}$	D	$\hat{H}=\sum_{i=1}^{n}\left(e^{2\hat{p}_i}+e^{-2\hat{p}_i}\right)\prod_{j\neq i}\frac{1}{(\hat{q}_i-\hat{q}_j)(\hat{q}_i+\hat{q}_j)}$