Schubert calculus and bosonic operators

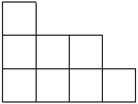
Gleb Nenashev

St Petersburg State University

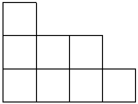
Beijing, July 9, 2024

We denote by $\mathbb{Y}$ the set of Young diagrams (partitions), i.e.,
$\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Y}$ s.t. $\lambda_1 \geq \ldots \geq \lambda_k \geq 0, \ \lambda_i \in \mathbb{Z}_{\geq 0}$. For example,

$$(4,3,1) \quad =$$



We denote by $\mathbb{Y}$ the set of Young diagrams (partitions), i.e., $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Y}$ s.t. $\lambda_1 \geq \dots \geq \lambda_k \geq 0,\ \lambda_i \in \mathbb{Z}_{\geq 0}$. For example,

$(4, 3, 1) \ =$ 

Define the vector space $\mathbb{QY}$ as formal finite sums of Young diagrams with rational coefficients, i.e.,

$$\mathbb{QY} := \left\{ \sum_{i=1}^{k} a_i \lambda^{(i)} :\ k \in \mathbb{N},\ a_i \in \mathbb{Q},\ \lambda^{(i)} \in \mathbb{Y} \right\}.$$

Operators for Young diagrams

S. V. Kerov defined the following two operators

$$U_z\lambda := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \mu=\lambda\cup(i,j)\in\mathbb{Y}}} (z+(j-i))\cdot\mu$$

and

$$D_{z'}\lambda := \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \mu=\lambda\setminus(i,j)\in\mathbb{Y}}} (z'+(j-i))\cdot\mu.$$

The number $j-i$ (column index minus row index) is the content of the box $\mu\setminus\lambda$ and $\lambda\setminus\mu$ resp.

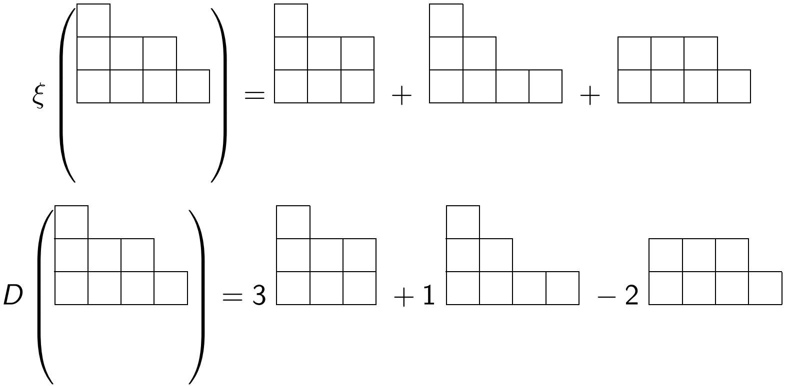
See A. Okounkov, *"SL(2) and z-measures."*

Define two linear "differential" operators on $\mathbb{QY}$. For a Young diagram $\lambda \in \mathbb{Y}$, we have

$$\xi(\lambda) := D_{z'+1}(\lambda) - D_{z'}(\lambda) = \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \mu=\lambda\setminus(i,j)\in\mathbb{Y}}} \mu;$$

and

$$D(\lambda) := D_0(\lambda) = \sum_{\substack{(i,j)\in\mathbb{N}^2 \\ \mu=\lambda\setminus(i,j)\in\mathbb{Y}}} (j-i)\cdot\mu.$$


ξ
=
+
+
D
= 3
+ 1
− 2

Key Lemma

For the empty diagram, we have $\xi(\emptyset) = D(\emptyset) = 0$, therefore we associate the empty diagram with $\emptyset = 1$. We say that q times the empty diagram is a constant.

Lemma (N.)

An element from $\mathbb{QY}$ is a constant if and only if both operators give zero, i.e.,

$$x \in \mathbb{Q} \iff \xi(x) = D(x) = 0.$$

We say that a bilinear map $\star : \mathbb{Q}\mathbb{Y}^2 \to \mathbb{Q}\mathbb{Y}$ is graded if

- for $\lambda, \mu \in \mathbb{Y}$, $\lambda \star \mu \in \mathbb{Q}\mathbb{Y}_{|\lambda|+|\mu|}$.

Corollary

There is at most one bilinear graded map $\star$ such that, for any $\lambda, \mu \in \mathbb{Y}$,

- $1 \star 1 = 1$, *where* 1 *is the empty diagram;*
- $\xi(\lambda \star \mu) = (\xi\lambda) \star \mu + \lambda \star (\xi\mu)$;
- $D(\lambda \star \mu) = (D\lambda) \star \mu + \lambda \star (D\mu)$.

Furthermore if there is such $\star$, then it satisfies commutative and associative properties.

Main theorem

Theorem (N.)

There is a unique bilinear graded map $\star$ *such that, for any* $\lambda, \mu \in \mathbb{Y}$*,*

- $1 \star 1 = 1$*, where* 1 *is the empty diagram;*
- $\xi(\lambda \star \mu) = (\xi\lambda) \star \mu + \lambda \star (\xi\mu)$*;*
- $D(\lambda \star \mu) = (D\lambda) \star \mu + \lambda \star (D\mu)$*.*

Furthermore it is given by

$$\lambda \star \mu = \sum_{\nu} c^{\nu}_{\lambda,\mu} \nu,$$

where $c^{\nu}_{\lambda,\mu}$ *are Littlewood-Richardson coefficients, i.e.,* $\star$ *is the multiplication for Schur functions.*

Jacobi-Trudi identity

$$h_\ell := \underbrace{\square\square\square\dots\square\square}_{\ell}$$

Theorem (Jacobi-Trudi identity)

For a partition $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0)$, *we have*

$$s_\lambda = \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \ldots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \ldots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \ldots & h_{\lambda_k} \end{bmatrix}$$

Proof

$$s_\lambda \stackrel{?}{=} \det{}_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We prove it by induction by $|\lambda| = \lambda_1 + \ldots + \lambda_k$.

Base case: $|\lambda| = 0$. We have $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 0$, therefore $s_\lambda = 1 = det_\lambda$.

Induction step: If $\xi(s_\lambda) = \xi(det_\lambda)$ and $D(s_\lambda) = D(det_\lambda)$, then by Key Lemma $s_\lambda = det_\lambda$.

Proof. Induction step: $\xi(s_\lambda) \overset{?}{=} \xi(det_\lambda)$

$$s_\lambda \overset{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_2+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We have

$$\xi(h_{\lambda_i-i+j}) = h_{(\lambda_i-1)-i+j},$$

then after combining by rows we get

$$\xi(\det_\lambda) = \sum_{\lambda'=\lambda\setminus(i,\lambda_i)\in\mathbb{Y}} \det_{\lambda'} = \sum_{\lambda'=\lambda\setminus(i,j)\in\mathbb{Y}} s_{\lambda'} = \xi(s_\lambda).$$

Proof. Induction step: $D(s_\lambda) \stackrel{?}{=} D(det_\lambda)$

$$s_\lambda \stackrel{?}{=} \det_\lambda := \det \begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & h_{\lambda_1+2} & \dots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & h_{\lambda_2+1} & \dots & h_{\lambda_w+k-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_k-k+1} & h_{\lambda_k-k+2} & h_{\lambda_k-k+3} & \dots & h_{\lambda_k} \end{bmatrix}$$

We have

$$D(h_{\lambda_i-i+j}) = (\lambda_i - i + j - 1)h_{\lambda_i-i+j-1} =$$
$$= (\lambda_i - i)h_{(\lambda_i-1)-i+j} + (j-1)h_{\lambda_i-i+(j-1)},$$

then

$$D(\det_\lambda) = \sum_{\lambda'=\lambda\setminus(i,\lambda_i)\in\mathbb{Y}} (\lambda_i - i)\det_{\lambda'} = \sum_{\lambda'=\lambda\setminus(i,j)\in\mathbb{Y}} (j-i)s_{\lambda'} = D(s_\lambda).$$

For the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, \ldots]$, the i-th divided differences operator is given by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$

Definition (Lascoux–Schützenberger)/
Theorem (Demazure and Bernstein–Gelfand–Gelfand)

For a permutation $w_0 = (n, n-1, \ldots, 1) \in S_n$, we define its Schubert polynomial as

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^1 \in \mathbb{Q}[x_1, x_2, \ldots].$$

For a permutation $w \in S_n$,

$$\partial_i \mathfrak{S}_w = \begin{cases} \mathfrak{S}_{ws_i} & \text{if } i \text{ is a } \textit{descent} \text{ of } w\text{, i.e., } \ell(ws_i) = \ell(w) - 1 \\ 0 & \text{if } i \text{ is an } \textit{ascent} \text{ of } w\text{, i.e., } \ell(ws_i) = \ell(w) + 1. \end{cases}$$

Theorem (Lascoux–Schützenberger)

For any $u \in S_{\mathbb{N}}$, its Schubert polynomial $\mathfrak{S}_u$ is well defined and $\mathfrak{S}_u$ is a homogeneous polynomial of degree $\ell(u)$.
The set $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ of all Schubert Polynomials forms a linear basis of $\mathbb{Q}[x_1, x_2, x_3, \ldots]$.

Therefore, we have unique coefficients $c^w_{u,v}, (u, v, w \in S_{\mathbb{N}})$ such that, for any $u, v \in S_{\mathbb{N}}$,

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_{\mathbb{N}}} c^w_{u,v} \mathfrak{S}_w.$$

Problem

Give a combinatorial interpretation of $c^w_{u,v}$.

Monk's rule (Monk)

For $u \in S_{\mathbb{N}}$ and $m \in \mathbb{N}$, we have

$$\mathfrak{S}_u \mathfrak{S}_{s_m} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots + x_m) = \sum_{a \leq m < b:\ \ell(ut_{a,b}) = \ell(u)+1} \mathfrak{S}_{ut_{a,b}},$$

where $t_{a,b}$ is a transposition of a and b.

Let $[a, b]$, $a < b \in \mathbb{N}$ be

$$\mathfrak{S}_u[a,b] = \begin{cases} \mathfrak{S}_{ut_{a,b}} & \text{if } \ell(ut_{a,b}) = \ell(u) + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Pieri's rule (Sottile)

For $u \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, we have

$$\mathfrak{S}_u \cdot h_k(x_1, x_2, \ldots, x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 \le i_2 \le \ldots \le i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k} \right) = $$

$$= \sum_{\substack{a_1 \le \ldots \le a_k \le m \\ m < b_1, \ldots, b_k \text{ are distinct}}} \mathfrak{S}_u [a_1 b_1][a_2 b_2] \cdots [a_k b_k]$$

and

$$\mathfrak{S}_u \cdot e_k(x_1, x_2, \ldots, x_m) = \mathfrak{S}_u \cdot \left(\sum_{i_1 < i_2 < \ldots < i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k} \right) = $$

$$= \sum_{\substack{a_1, \ldots, a_k \le m \text{ are distinct} \\ m < b_1 \le \ldots \le b_k}} \mathfrak{S}_u [a_1 b_1][a_2 b_2] \cdots [a_k b_k]$$

Operator D for Schubert polynomials

Let

$$D = \sum_{i \in \mathbb{N}} \frac{\partial}{\partial x_i}$$

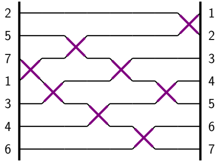
be the sum of all partial derivatives. Hence, we have *Leibniz product rule* for D:

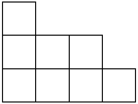
$$D(\mathfrak{S}_u \mathfrak{S}_v) = (D\mathfrak{S}_u)\mathfrak{S}_v + \mathfrak{S}_u(D\mathfrak{S}_v).$$

Theorem (Hamaker–Pechenik–Speyer–Weigandt)

For any $u \in S_{\mathbb{N}}$,

$$D\mathfrak{S}_u = \sum_{k \in \mathbb{N}:\ \ell(s_k u) := \ell(u) - 1} k \mathfrak{S}_{s_k u}.$$





Theorem

For a Grassmannian permutation $w \in S_{\mathbb{N}}$ of a descent m, we have

$$\mathfrak{S}_w = s_{\lambda(w)}(x_1, x_2, \ldots, x_m),$$

where $\lambda(w) = (w_m - m, w_{m-1} - m + 1, w_{m-2} - m + 2, \ldots)$.

Theorem

For a Grassmannian permutation $w \in S_{\mathbb{N}}$ of a descent m, we have

$$D\mathfrak{S}_w = D_m s_{\lambda(w)}(x_1, x_2, \ldots, x_m).$$

Stabilities

Let τ be a shift defined by

$$\tau w(i) = w(i-1)+1,\ i \in \mathbb{Z} \quad \text{for } w \in S_{\mathbb{Z}}.$$

For $w \in S_{\mathbb{N}}$, its *Stanley symmetric function* for $w \in S_{\mathbb{N}}$ is given by

$$\mathcal{F}_w(x_1, x_2, \ldots) := \lim_{k \to +\infty} \mathfrak{S}_{\tau^k w}(x_1, x_2, \ldots) \in \Lambda_+.$$

For $w \in S_{\mathbb{Z}}$, its *back-stable Schubert polynomial* is given by

$$\overleftarrow{\mathfrak{S}}_w(x_i, i \in \mathbb{Z}) := \lim_{k \to +\infty} \mathfrak{S}_{\tau^k w}(x_{1-k}, x_{2-k}, \ldots) \in \Lambda_- \otimes \mathbb{Q}[x_i, i \in \mathbb{Z}].$$

Theorem (Edelman–Greene)

For any $w \in S_{\mathbb{N}}$,

$$\mathcal{F}_w(x_1, x_2, \ldots) = \sum_{\lambda} a_{\lambda,w} s_\lambda(x_1, x_2, \ldots),$$

where $a_{\lambda,w} \in \mathbb{Z}_{\geq 0}$ is the number of increasing Young tableux of the shape λ, whose reads are reduced words of w^{-1}.

3			
2	4	5	
1	3	4	6

$$s_3 s_2 s_4 s_5 s_1 s_3 s_4 s_6 = (2, 5, 7, 1, 3, 4, 6)^{-1}$$

See Lam–Lee–Shimozono for another interpretation of $a_{\lambda,w}$.

Theorem (Lam–Lee–Shimozono)

The set $\{\overleftarrow{\mathfrak{S}}_u, u \in S_{\mathbb{Z}}\}$ *is a linear basis of the ring* $\mathbb{B} := \Lambda_- \otimes \mathbb{Q}[x_i, i \in \mathbb{Z}]$.

Therefore, for any $u, v \in S_{\mathbb{Z}}$,

$$\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v = \sum_{w \in S_{\mathbb{Z}}} c^w_{u,v} \overleftarrow{\mathfrak{S}}_w.$$

For any $u, v, w \in S_{\mathbb{Z}}$, we have

$$c^w_{u,v} = c^{\tau w}_{\tau u, \tau v} = c^{\hat{w}}_{\hat{u},\hat{v}}, \quad \text{where } \hat{u}(i) = -u(-i).$$

Furthermore, for any $u, v, w \in S_{\mathbb{N}}$, $c^w_{u,v}$ is equal to the corresponding structure constant for Schubert polynomials.

Operators ξ and D on $\overleftarrow{\mathfrak{S}}$

$$\xi \overleftarrow{\mathfrak{S}}_u := \sum_{k\in\mathbb{Z}:\ \ell(s_k u)=\ell(u)-1} \overleftarrow{\mathfrak{S}}_{s_k u};$$

$$D \overleftarrow{\mathfrak{S}}_u := \sum_{k\in\mathbb{Z}:\ \ell(s_k u)=\ell(u)-1} k \overleftarrow{\mathfrak{S}}_{s_k u}.$$

Proposition (N.)

For any $u, v \in S_{\mathbb{Z}}$, we have:

$$\xi(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\xi \overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(\xi \overleftarrow{\mathfrak{S}}_v);$$

$$D(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (D \overleftarrow{\mathfrak{S}}_u)\overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u(D \overleftarrow{\mathfrak{S}}_v).$$

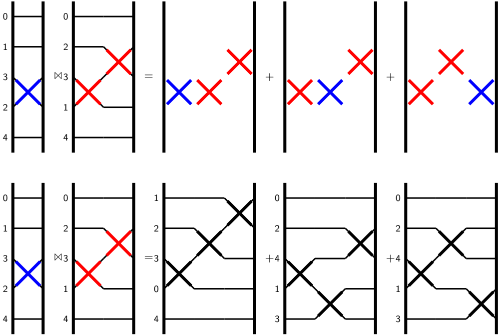
Corollary

Given a pair of permutations $u, v \in S_{\mathbb{Z}}$, the following holds:

$$\binom{\ell(u)+\ell(v)}{\ell(v)} |\mathcal{R}(u)||\mathcal{R}(v)| = \sum_{w \in S_{\mathbb{Z}}} c_{u,v}^{w} \, |\mathcal{R}(w)|,$$

where $\mathcal{R}(u)$ is the set of reduced words of u.

Example



$$\overleftarrow{\mathfrak{S}}_{(01324)}\overleftarrow{\mathfrak{S}}_{(02314)} = \overleftarrow{\mathfrak{S}}_{(12304)} + \overleftarrow{\mathfrak{S}}_{(02413)} \qquad \binom{3}{2}\cdot 1\cdot 1 = 1+2$$

$$\mathfrak{S}_{(1324)}\mathfrak{S}_{(2314)} = \mathfrak{S}_{(2413)} \qquad \binom{3}{2}\cdot 1\cdot 1 \neq 2$$

Theorem (N.)

Given an operator η satisfying Leibniz product rule such that

$$\eta \overleftarrow{\mathfrak{S}}_u \in span\{\overleftarrow{\mathfrak{S}}_{s_k u} : \ \ell(s_k u) = \ell(u) - 1\}.$$

Then η is a linera combination of ξ and D.

Criterion (postive)

Theorem A (N.)

There is a unique graded bilinear map $\star$ such that, for any $u, v \in S_{\mathbb{Z}}$,

- $id \star id = id$;
- $\xi(u \star v) = (\xi u) \star v + u \star (\xi v)$;
- $D(u \star v) = (Du) \star v + u \star (Dv)$;
- $u \star v \in \mathbb{Z}_{\geq 0} S_{\mathbb{Z}}$.

Furthermore it is given by

$$u \star v = \sum_{w} c_{u,v}^{w} w,$$

where $c_{u,v}^{w}$ are structure constants for back-stable Schubert polynomials.

Criterion (supp; weak)

Let $S_{\mathbb{Z},k}$ be the set of permutations s.t. that have s_k in reduced words. Consider any function $f : S_{\mathbb{Z}}^2 \to \mathbb{Z}$ such that, for any $(u, v) \neq (id, id)$, either u or v or both have $s_{f(u,v)}$ in reduced words.

Theorem B (N.)

There is a unique graded bilinear map $\star$ such that, for any $u, v \in S_{\mathbb{Z}}$,

- $id \star id = id$;
- $\xi(u \star v) = (\xi u) \star v + u \star (\xi v)$;
- $D(u \star v) = (Du) \star v + u \star (Dv)$;
- $u \star v \in \mathbb{Q}S_{\mathbb{Z},f(u,v)}$.

Furthermore it is given by

$$u \star v = \sum_{w} c_{u,v}^{w} w,$$

where $c_{u,v}^{w}$ are structure constants for back-stable Schubert polynomials.

Proof of the Monk's rule

$D = \sum_k k d_k$ and $\mathcal{M}_z = \sum_{a \le z < b} [a, b]$.
We know

$$d_k(\overleftarrow{\mathfrak{S}}_u[a, b]) = (d_k \overleftarrow{\mathfrak{S}}_u)[a, b] \text{ if } \{u(a), u(b)\} \neq \{k, k+1\},$$

hence

$$D(\overleftarrow{\mathfrak{S}}_u \mathcal{M}_z) =$$

$$= (D\overleftarrow{\mathfrak{S}}_u)\mathcal{M}_z + \left(\sum_{\substack{k \in \mathbb{Z}: \\ u^{-1}(k) \le z < u^{-1}(k+1)}} k - \sum_{\substack{k' \in \mathbb{Z}: \\ u^{-1}(k') > z \ge u^{-1}(k'+1)}} k' \right) \cdot \overleftarrow{\mathfrak{S}}_u =$$

$$= (D\overleftarrow{\mathfrak{S}}_u)\mathcal{M}_z + z \cdot \overleftarrow{\mathfrak{S}}_u = (D\overleftarrow{\mathfrak{S}}_u)\mathcal{M}_z + \overleftarrow{\mathfrak{S}}_u \cdot (D\overleftarrow{\mathfrak{S}}_{s_z}).$$

■

Bosonic operators

Define the sequence of *bosonic* operators

- $\rho^{(1)} := \xi$;
- $\rho^{(k+1)} := \frac{[\rho^{(k)}, D]}{k} = \frac{\rho^{(k)} \cdot D - D \cdot \rho^{(k)}}{k}$.

Corollary

For any $k \in \mathbb{N}$ and $u, v \in S_{\mathbb{Z}}$, we have

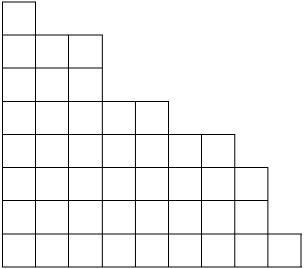
$$\rho^{(k)}(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = (\rho^{(k)} \overleftarrow{\mathfrak{S}}_u) \overleftarrow{\mathfrak{S}}_v + \overleftarrow{\mathfrak{S}}_u (\rho^{(k)} \overleftarrow{\mathfrak{S}}_v).$$

Theorem (N.)

Operators $\rho^{(k)}, k \in \mathbb{N}$ commute pairwise.

Theorem (dual Murnaghan-Nakayama)

$$\rho^{(k)} s_\lambda := \sum_{\substack{\mu:\ \mu\subset\lambda,\ |\mu|=|\lambda|-k, \\ \lambda\setminus\mu \textit{ is a border strip}}} (-1)^{ht(\lambda\setminus\mu)-1} s_\mu .$$



Theorem (N.)

$$\rho^{(k)}\overleftarrow{\mathfrak{S}}_u := \sum_{w\ s.t.\dots} \pm\overleftarrow{\mathfrak{S}}_w.$$

For a partition $\lambda \in \mathbb{Y}$ we define operator ξ^λ as

$$\xi^\lambda := \sum_\mu \frac{\chi^\lambda_\mu}{z_\mu} \rho^{(\mu_1)} \dots \rho^{(\mu_k)}.$$

Theorem

The operators $\xi^\lambda,\ \lambda \in \mathbb{Y}$ commute pairwise and satisfy the following three properties

$$\xi^\nu \xi^\mu = \sum_\lambda \mathcal{LR}^\lambda_{\nu,\mu} \xi^\lambda;$$

$$\xi^\lambda(\overleftarrow{\mathfrak{S}}_u \overleftarrow{\mathfrak{S}}_v) = \sum_{\nu,\mu} \mathcal{LR}^\lambda_{\nu,\mu} (\xi^\nu \overleftarrow{\mathfrak{S}}_u)(\xi^\mu \overleftarrow{\mathfrak{S}}_v);$$

$$\xi^\lambda \overleftarrow{\mathfrak{S}}_w = \sum_{\substack{\ell(u)=|\lambda| \\ \ell(u^{-1}w)=\ell(w)-|\lambda|}} a_{\lambda,u} \overleftarrow{\mathfrak{S}}_{u^{-1}w}.$$

Thank You!