Quadratic Algebras, Schubert Calculus and Small Quantum Cohomology of Flag Varieties (of type A)

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Contents

Motivations

ⓐ In a few words, classical Schubert and Grothendieck calculi are developed to solve problems in Enumerative Algebraic Geometry and Algebraic Combinatorics.

For example the algebraic description of Schubert (or Grothendieck, Chern-Schwartz-Macpherson, Severi, . . .) classes in the cohomology theory (or K-theory) of complete or parabolic flag varieties, Heisenberg varieties, . . . (of type A in this talk).

The basic idea is to study the generating functions of these objects and their properties using algebraic methods.

For example, let NC_n (resp. NH_n) be the Nil-Coxeter (resp. Nil-Hecke algebra) defined over $\mathbb{Z}$, with generators $e_1, \cdots, e_{n-1}$, and relations

$$\begin{aligned} e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} & \qquad \text{(Coxeter relations)}, \\ e_i e_j = e_j e_i \quad \text{if } |i-j| > 1 & \quad \text{(locality relations)}, \\ e_i^2 = 0, \quad \text{resp. } e_i^2 = \beta e_i, & \\ & \text{with parameter } \beta \text{ in Nil-Hecke case.} \end{aligned}$$

Theorem (S. Fomin, AK)

Let NC_n *(resp.* NH_n*) be the nil-Coxeter (resp. nil-Hecke algebra). Define* $A_i(x) = \prod_{j=1}^{i}(1 + xe_{n-i})$*.*

Then the Cauchy kernel $C_n(x_1, \cdots, x_n) := \prod_{i=1}^{n-1} A_i(x_i)$ *admits the following decompositions:*

$$C_n(x_1, \cdots x_n) = \sum_{w \in S_n} \mathfrak{S}_w(x_1, \ldots, x_n) \cdot e_w \quad \textit{in } \mathrm{NC}_n;$$
$$= \sum_{w \in S_n} G_w(x_1, \ldots, x_n) \cdot e_w \quad \textit{in } \mathrm{NH}_n.$$

where $\mathfrak{S}_w$ *is the Lascoux-Schützenberger Schubert polynomial and* G_w *the Grothendieck polynomial.*

ⓑ Truncated Dunkl operators

Theorem (Charles Dunkl, Eric Opdam)

Consider a finite complex reflection group

$$(W, S, V), \qquad |S| = \ell.$$

Then the algebra generated by mutually commutative truncated Dunkl operators $D_1, \cdots D_\ell$ is isomorphic to the coinvariant algebra of that group:

$$A_{W,S} := \mathbb{C}[x_1, \cdots, x_\ell]/\langle J_1, \cdots, J_\ell\rangle \cong \mathbb{C}[D_1, \ldots, D_\ell].$$

where $J_1, \ldots, J_\ell$ denote the fundamental invariants of the group (W, S, V).

In particular, in the Weyl group case, the coinvariant algebra is the cohomology ring $H^*(G/B, \mathbb{Q})$ of the flag variety G/B, so the truncated Dunkl operators define a system of generators of G/B.

Note that truncated Dunkl operators have natural extensions for trigonometic, elliptic, (small) quantum theories (e.g. K-theory, equivariant and elliptic cohomology).

The main objective of this talk is to consider generalized Dunkl elements in certain (non-homogeneous, non-commutative) quadratic algebras. We restrict to type A.

Some quadratic algebras

Let R be a commutative ring, $\operatorname{char} R = 0$. We assume that all parameters below belong to R.

Next, let us fix an integer $n \geq 2$. Consider the set of variables $\{u_{ij} \,|\, 1 \leq i, j \leq n\}$ and the corresponding (unital associative) algebra

$$\mathbb{A}_n = R\langle u_{ij} \,|\, 1 \leq i, j \leq n \rangle.$$

To distinguish the role of element u_{ii} and the elements u_{ij} with $i \neq j$, we set

$$z_i := u_{ii}.$$

Definition (Dunkl elements)

$$\theta_i := z_i + \sum_{j \neq i} u_{ij} \in \mathbb{A}_n.$$

Let us compute commutators of Dunkl elements:

$$
\begin{aligned}
[\theta_i, \theta_j] &= \theta_i\theta_j - \theta_j\theta_i \\
&= [z_i + u_{ij}, z_j + u_{ji}] \quad \text{zero curvature} \\
&\quad + [z_i + z_j, u_{ij}] \quad \text{translation invariance} \\
&\quad + \left[u_{ij}, \sum_{k=1}^{n} z_k\right] \quad \text{homogeneity} \\
&\quad + \sum_{k \neq i,j} \left\{ \begin{array}{l} [u_{ik}, u_{jk}] + [u_{ij}, u_{jk}] + [u_{ik}, u_{ji}] + \\ +[z_i, u_{jk}] + [z_k, u_{ij}] + [u_{ik}, z_j] \end{array} \right\} \\
&\quad \text{dynamical Yang-Baxter relns.} \\
&\quad + \sum_{k<l} \left\{ [u_{ik}, u_{jl}] + [u_{il}, u_{jk}] \right\}. \\
&\quad \text{Manin's nondiagonal relations}
\end{aligned}
$$

Consider the *six-term relations algebra*

$$6\mathrm{T}_n := \mathbb{A}_n / \langle \text{above terms} \rangle.$$

Clearly, in $6\mathrm{T}_n$, the Dunkl elements θ_i generate a commutative subalgebra.

Problem

Describe the group which is "FRT-dual" to $6\mathrm{T}_n$.

Note that

$$6\mathrm{T}_n / \langle z_1, \ldots, z_n, [u_{ij}, u_{kl}]\,(\forall i, j, k, l)\rangle$$

is isomorphic to the polynomial algebra

$$\mathbb{C}[u_{ij} \,|\, 1 \leq i \neq j \leq n].$$

We leave to the audience the following questions:

(a) What is the subalgebra of $6\mathrm{T}_n / \langle z_1, \ldots, z_n\rangle$ generated by the "simple roots" $u_{i,i+1}$ $(i = 1, \cdots, n-1)$?

(b) Same question for the simple roots $u_{i,i+1}$ $(i = 1, \cdots, n-1)$ and $-u_{1n}$.

Calogero-Moser-type integrable systems

If we let the generators of $6\mathrm{T}_n$ act as

$$z_i = \frac{\mathrm{d}}{\mathrm{d}x_i}, \qquad u_{ij} = \frac{1 - s_{ij}}{x_i - x_j},$$

then the θ_i act as Dunkl operators for the *Calogero-Moser integrable system*. The following *exchange relation* holds:

$$z_i u_{ij} = u_{ij} z_j - \frac{z_i - z_j + u_{ij}}{x_i - x_j}.$$

Contrary to this "honest" definition of Dunkl operators, we introduce (equivariant) *truncated Dunkl elements*. For this goal we need a different algebra.

Generalized Fomin-Kirillov algebras

Let $\beta \in R$. The generalized FK algebra $3\mathrm{T}_{n,\beta}$ is the quotient of $6\mathrm{T}_n$ by the 2-sided ideal generated by:

(Locality) $[z_i, u_{jk}] = [u_{ij}, u_{kl}] = 0$, for distinct i, j, k, l.

(Factorizibility) For distinct i, j, k we have

$$u_{ij}u_{jk} = u_{jk}u_{ik} - u_{ik}u_{ji}, \qquad u_{jk}u_{ij} = u_{ik}u_{jk} - u_{ji}u_{ik}.$$

(Unitarity) $u_{ij} + u_{ji} = \beta$.

(Exchange relation) There exists $d \in R$ such that

$$z_i u_{ij} + u_{ji} z_j = d, \qquad \text{if } i < j.$$

(Central elements) $q_{ij} \in R$, where

$$q_{ij} := u_{ij}^2 - \beta u_{ij} \qquad = -u_{ij}u_{ji}.$$

Elements $\{u_{ij}\}$ satisfying locality, factorizability and unitarity conditions have the following properties:

- (quantum-Yang-Baxter relations)

$$u_{ij}u_{ik}u_{jk} - u_{jk}u_{ik}u_{ij} = [u_{ik}, q_{ij}]$$
$$u_{ij}u_{ik}u_{jk} - u_{jk}u_{ik}u_{ij} = [q_{jk}, u_{ik}]$$

  for all i, j, k distinct (follows from unitarity). Hence we obtain the celebrated Kohno-Drinfeld relations

$$[q_{ij} + q_{jk}, q_{ik}] = 0.$$

- (Coxeter relations)

$$u_{ij}u_{jk}u_{ij} - u_{jk}u_{ij}u_{jk} = u_{ik}q_{ij} - q_{jk}u_{ik}.$$

- (4-term relations of degree 3)

$$u_{il}u_{jk}u_{ik}+u_{ik}u_{il}u_{jl}-\beta u_{il}u_{jk} = u_{jk}u_{ik}u_{il}+u_{il}u_{jl}u_{jk}-\beta u_{ik}u_{jl}.$$

  Hint: for the proof, use associativity to write $u_{ij}u_{jk}u_{jl}$ in two ways.

- ("cyclic" relations) Let $3 \leq p \leq n$. Define

$$C_{p,n} = u_{1n}(u_{2n} \cdots u_{pn})u_{n1} + u_{2n}(u_{3n} \cdots u_{pn}u_{n1})u_{n2} + \ldots + u_{pn}(u_{n1} \cdots u_{n,p-1})u_{np}.$$

Then in the algebra $3\mathrm{T}_{n,\beta}$, one has:

$$C_{p,n} = \sum_k \Big(\prod_{a=p-1}^{1} u_{ap} \Big) q_{pn} \prod_{a=n}^{p} u_{ak}.$$

These identities play an essential role in the study of the structure of the algebra $3\mathrm{T}_{n,\beta}$.

Truncated Dunkl elements

The "equivariant" Dunkl elements are given by

$$\theta_i = z_i + \sum_{j \neq i} u_{ij}.$$

The corresponding *truncated Dunkl elements* are:

$$\bar{\theta}_i = \sum_{j \neq i} u_{ij}.$$

For example, if $n = 4$,

$$\bar{\theta}_1 = u_{12} + u_{13} + u_{14}, \qquad \bar{\theta}_2 = u_{21} + u_{23} + u_{24}$$
$$\bar{\theta}_3 = u_{31} + u_{32} + u_{34}, \qquad \bar{\theta}_4 = u_{41} + u_{42} + u_{43}.$$

Define the *quantum elementary symmetric polynomial*:

$$e_k^{\mathrm{qu}}(x_1,\ldots,x_n) = \sum_{\substack{I,J\subset[1,n],\ I\cap J=\emptyset \\ |I|=|J|=k}} \left(\prod_{i\in I, j\in I} q_{i_\alpha, j_\beta}\right)\times \\ \times\, e_{n-2k}(\{x_j \mid j\in[1,n]\setminus I\cup J\}).$$

Theorem (AK, Fomin)

$$e_k^{qu}(\bar{\theta}_1,\cdots,\bar{\theta}_n) = 0.$$

Comment

We view a polynomial $P(x_1,\ldots,x_n)$ as an *integral of motion* in the algebra $3\mathrm{T}_{n,\beta}$, if $P(\bar{\theta}_1,\cdots,\bar{\theta}_n)=0$.

Classical algebra $3T_n^{(0)}$

Set $\beta = 0$ and consider the quotient

$$3T_n^{(0)} := 3T_{n,0} / \langle z_1, \ldots, z_n, \{u_{ij}^2 \mid i \neq j\} \rangle.$$

Note, $3T_n^{(0)}$ has $\binom{n}{2}$ generators u_{ij} $(1 \leq i < j \leq n)$ and $\binom{n}{2} + 2\binom{n}{3} + 3\binom{n}{4}$ quadratic relations:

$$u_{ij}^2 = 0 \qquad \text{if } i < j,$$

$$\left.\begin{aligned} u_{jk}u_{ij} &= u_{ik}u_{jk} + u_{ij}u_{ik}, \\ u_{ij}u_{jk} &= u_{jk}u_{ik} + u_{ik}u_{ij}, \end{aligned}\right. \qquad \text{if } i < j < k.$$

The subalgebra $\langle u_{i,i+1} \mid i = 1, \cdots, n-1 \rangle$ is isomorphic to the Nil-Coxeter algebra NC_n.

It is known that

$$\begin{aligned} \mathrm{Hilb}(3\mathrm{T}_3^{(0)}, t) &= [2]_t^2 \cdot [3]_t, & \dim 3\mathrm{T}_3^{(0)} &= 12, \\ \mathrm{Hilb}(3\mathrm{T}_4^{(0)}, t) &= [2]_t^2 \cdot [3]_t^2 \cdot [4]_t^2, & \dim 3\mathrm{T}_4^{(0)} &= 24^2, \\ \mathrm{Hilb}(3\mathrm{T}_5^{(0)}, t) &= [4]_t^4 \cdot [5]_t^2 \cdot [6]_t^4, & \dim 3\mathrm{T}_5^{(0)} &< \infty. \end{aligned}$$

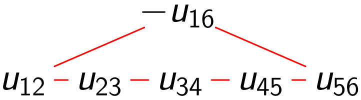
Problem

1. *Compute the Hilbert series* $\mathrm{Hilb}(3\mathrm{T}_n^{(0)}, t)$ *for* $n \geq 6$.
2. *Find a monomial basis in the algebra* $3\mathrm{T}_n^{(0)}$, *for* $n \geq 6$.
3. *Prove/disprove that* $\dim 3\mathrm{T}_6^{(0)} = +\infty$.

A strange observation: $\mathrm{Hilb}(E_8, t)/\,\mathrm{Hilb}(3\mathrm{T}_0^{(5)}, t) \in \mathbb{Z}[t]$.

Let $3T_n^{(\text{aff})}$ be the subalgebra of $3T_n^{(0)}$ generated by

$$u_{i,i+1} \quad (1 \leq i \leq n-1), \qquad -u_{1n}.$$



Theorem (Y. Bazlov, AK)

The algebra $3T_n^{(\text{aff})}$ (if $n \geq 2$) is finite dimensional and its Hilbert polynomial equals

$$\text{Hilb}(3T_n^{(\text{aff})}, t) = [n]_t \prod_{i=1}^{n-1} \frac{1 - t^{i(n-i)}}{1-t}.$$

Cohomology of flag varieties and $3T_n^{(0)}$

Theorem [S. Fomin, AK, 1992]

$$\mathbb{Z}[\theta_1, \cdots, \theta_n] \cong H^*(\mathrm{Fl}_n, \mathbb{Z}),$$

where Fl_n denotes the complete flag variety of $GL(n)$.

Proof consists of two parts.

Part I. $\qquad e_k(\theta_1, \cdots, \theta_n) = 0 \qquad$ if $k \geq 1$.
This follows from the cyclic relations mentioned above.

Therefore, the following natural map is an epimorphism:

$$\mathbb{Z}[\theta_1, \cdots, \theta_n] \to H^*(\mathrm{Fl}_n, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n]/\langle e_k(t_1, \ldots, t_n)\rangle$$

$$\theta_i \mapsto t_i.$$

Part II (Bruhat representation)
We define an action of $3\mathrm{T}_n^{(0)}$ on the group ring of the symmetric group S_n:

$$u_{ij} \circ w = \begin{cases} ws_{ij} & \text{if } \ell(ws_{ij}) = \ell(w) + 1, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{where } w \in S_n.$$

To see that this is an action, a computation in S_3 implies that, for distinct i, j, k,

$$(u_{ij}u_{jk} - u_{jk}u_{ik} - u_{ik}u_{ij}) \circ w = 0 \qquad \forall w \in S_n.$$

Lemma

Under the above epimorphism

$$\mathbb{Z}[\theta_1, \cdots, \theta_n] \to H^*(\mathrm{Fl}_n, \mathbb{Z}),$$

the element θ_i corresponds to multiplication by t_i in $H^*(\mathrm{Fl}_n, \mathbb{Z})$.

Hint: follows from Monk's formula, see e.g. Macdonald's book on Schubert Calculus.

This completes the proof of the Theorem.

Theorem (AK, Maeno)

There exist unique polynomials

$$P_w(t_1, \cdots, t_n) \in \mathbb{Z}[t_1, \cdots, t_n]$$

such that

$$\mathit{support}(P_w) \subset (n-1, \cdots, 1, 0),$$

and, under Bruhat's representation,

$$P_w(\theta_1, \cdots, \theta_n) \circ 1 = w \in S_n.$$

Corollary

$$P_w(t_1, \ldots, t_n) = \mathfrak{S}_w(t_1, \ldots, t_n),$$

where $\mathfrak{S}_w$ is the Lascoux-Schützenberger Schubert polynomial.

Corollary

$$\mathfrak{S}_u(\theta_1, \cdots, \theta_n) \circ v = \sum c_{u,v}^w w \in \mathbb{Z}[S_n],$$

where $c_{u,v}^w$ $(u, v, w \in S_n)$ are the structure constants in

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_{w \in S_n} c_{u,v}^w \mathfrak{S}_w.$$

Conjecture (S. Fomin, AK)

If $w \in S_n$ *and* $m < n$, $\mathfrak{S}_w(\theta_1, \cdots, \theta_m, 0, \ldots, 0)$, *lies in* $\mathrm{Cone}^+(3\mathsf{T}_{\mathrm{n}}^{(0)})$ *(i.e. can be written as a positive combination of monomials).*

In particular, we expect

$$s_\lambda(\theta_1, \cdots, \theta_m, 0, \ldots, 0) \in \mathrm{Cone}^+(3\mathsf{T}_{\mathrm{n}}^{(0)}).$$

Problem

Describe the cones of positive elements in $3\mathsf{T}_n^{(0)}$.

Comments

According to Borel's theorem,

- $H^*(\mathsf{Fl}_n, \mathbb{Z})$ is generated by Chern classes of tautological linear bundles $\mathsf{Ch}(L_i)$ associated to simple roots;
- the map $\theta_i \mapsto \mathsf{Ch}(L_i)$ is a bijection.

We view the θ_i as non-commutative Chern classes of tautological vector bundles over flag varieties.

According to Grothendieck, any finite-dimensional vector bundle over Fl_n is isomorphic to $\bigoplus_{\lambda \vdash n} L_\lambda$.

By BBW, we can view $s_\lambda(\theta_1, \cdots, \theta_m, 0, \ldots, 0)$ as a non-commutative version of the irreducible highest weight representation V_λ. We expect: L_λ is positive if

$$s_\lambda(\theta_1, \cdots, \theta_m, 0, \ldots, 0) \in \mathrm{Cone}^+(3\mathsf{T}_{\mathrm{n}}^{(0)}).$$

Multiplicative Dunkl elements ($\beta \neq 0$)

For $x \in R$, define

$$h_{ij}(x) := 1 + x u_{ij}.$$

Then $h_{ij}(x) h_{ij}(y) = h_{ij}(x + y - \beta xy)$, so that

$$h_{ij}^{-1}(x) = h_{ij}\Big(-\frac{x}{1+\beta x}\Big).$$

Consider the localization

$$\widetilde{3\mathrm{T}}_{n,\beta} = 3\mathrm{T}_{n,\beta}\left[\left\{\frac{1}{1+\beta x - q_{ij}x^2}\right\}_{i<j}\right]$$

Definition: multiplicative Dunkl elements

Fix generic $x \in R$ and define

$$\Theta_i := \left(\prod_{j=1}^{i-1} h_{ij}^{-1}(x) \right) \left(\prod_{j=n}^{i+1} h_{ij}(x) \right) \in \widetilde{3\mathrm{T}}_{n,\beta}.$$

Proposition (AK)

$$[\Theta_i, \Theta_j] = 0 \qquad \textit{for all } i = 1, \cdots, n.$$

The proof relies on the constant YBE, due to C.N. Yang and Young:

$$h_{ij}(x) h_{ik}(x) h_{jk}(x) = h_{jk}(x) h_{ik}(x) h_{ij}(x).$$

Set $\Theta_J := \prod_{j \in J} \Theta_j$.

Theorem (AK)

$$\sum_{\substack{J \subset [1,n] \\ |J|=k}} \prod_{\substack{i \in J \\ j \notin J}} (1 + \beta x - x^2 q_{ij}) \Theta_J = \begin{bmatrix} n \\ k \end{bmatrix}_{1+\beta x}.$$

where we note that $(1 + \beta x - x^2 q_{ij}) h_{ij}^{-1}(x) \in 3\mathrm{T}_n^{(\beta)}$.

The proof relies on the relations

$$h_{ij}(x) h_{jk}(y) = h_{jk}(y) h_{ik}(x) + h_{ik}(y) h_{jk}(x) - h_{ik}(x + y + \beta xy)$$

with i, j, k distinct.

Elliptic functions and elliptic algebra $3T_n$

Let $q \in \mathbb{C}$ with $|q| < 1$. We consider the odd (Riemann) theta function $\theta : \mathbb{C} \to \mathbb{C}$, namely

$$\theta(t) := (t; q)_\infty (q/t; q)_\infty, \qquad \theta(-t) = -\theta(t)$$

which should not be confused with the Dunkl element θ_i.

It satisfies the 4-term relation:

$$\theta(x \pm y)\theta(z \pm t) = \theta(x \pm t)\theta(y \pm z) + \theta(x \pm z)\theta(y \pm t).$$

Here we use the standard notation

$$\theta(x \pm y) := \theta(x + y)\theta(x - y).$$

Next, we define Kronecker's sigma function to be

$$\sigma_\lambda(t) = \frac{\theta(t-\lambda)}{\theta(t)\theta(-\lambda)}$$

(we skip the constant $\theta'(0)$ from the numerator of RHS).

Given a tuple $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ (called *dynamical parameters*), we define the following operator on the space of meromorphic functions of n variables:

$$u_{ij} := \sigma_{\lambda_i - \lambda_j}(x_i - x_j) K_{ij}$$

where K_{ij} is the permutation operator.

The operator u_{ij} satisfies the following relations:

$$u_{ij}u_{jk} = u_{jk}u_{ik} + u_{ik}u_{ij}, \quad \text{if } i < j < k, \text{or } i > j > k$$

associative Yang-Baxter relations,

$$u_{ij} + u_{ji} = 0, \qquad \text{unitarity},$$

$$u_{ij}^2 = \wp(x_i - x_j) + \wp(\lambda_i - \lambda_j),$$

(quasiconstant + constant)

$$[u_{ij}, u_{kl}] = 0, \quad \text{if } i, j, k, l \text{ are distinct.}$$ (lo

It inspires the following definition of an *elliptic* 3-term algebra.

Definition: elliptic three-term relations algebra

We define $E3T_n$ as the algebra over $\mathbb{Q}(\{q_{ij}, \psi_{ij}\}_{1 \le i \ne j \le n})$ with generators u_{ij} $(1 \le i \ne j \le n)$ subject to:

1. $[u_{ij}, u_{kl}] = 0$ and $q_{ij} u_{kl} = u_{kl} q_{ij}$ if i, j, k, l are distinct.
2. $\psi_{ij} u_{jk} = u_{jk} \psi_{ik}$ if i, j, k are distinct.
3. $u_{ij} u_{jk} = u_{jk} u_{ik} + u_{ik} u_{ij}$ if $i < j < k$.
4. $u_{ji} = -u_{ij}, \quad q_{ij} = q_{ji}, \quad \psi_{ij} = \psi_{ji}.$
5. $u_{ij}^2 = \psi_{ij} + q_{ij}$; moreover $\exists \xi_{ij} = -\xi_{ji} \in E3T_n$ such that $\xi_{ij}^2 - \psi_{ij}$ is central $(i < j)$.

Note: in the representation on meromorphic functions, one can take

$$\xi_{ij} := \sigma_{\mu_i - \mu_j}(x_i - x_j)$$

for free parameters $\mu_1, \dots, \mu_n$, and then we have

$$\xi_{ij}^2 - \psi_{ij} = \sigma_{\mu_i - \mu_j}(\lambda_i - \lambda_j).$$

Theorem (AK)

The elements

$$R_{ij} := \xi_{ij} + u_{ij} \in \mathrm{E3T}_n, \qquad i \neq j$$

are invertible and satisfy the (constant) Yang-Baxter relations

$$R_{ij} R_{ik} R_{jk} = R_{jk} R_{ik} R_{ij} \qquad \text{if} \qquad i < j < k$$

Extend $E3T_n$ by elements $L_1, \ldots, L_n$ such that, for all $i < j$,

$$L_j L_i R_{ij} = R_{ij} L_i L_j.$$

Corollary (AK)

The following multiplicative Dunkl elements pairwise commute:

$$\Theta_i := \left(\prod_{a=i-1}^{1} R_{ia}^{-1} \right) L_i \left(\prod_{a=n}^{i+1} R_{ia} \right)$$

If $L_i = 1$, the Θ_i are "truncated Ruijsenaars-Macdonald operators" and we have the following explicit formula for quantum elementary symmetric polynomials:

$$e_n^{\mathrm{qu}}(\Theta_1, \cdots, \Theta_n) = \begin{cases} \displaystyle\sum_{\substack{A \subset [1,n] \\ |A| = n/2}} \prod_{\substack{i \in A \\ j \notin A}} \psi_{ij} & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}. \end{cases}$$

If $L_i = T_{i,q}$ (Ruijssenaars shift operator) then the Θ_i are known as *Ruijsenaars-Macdonald operators*.

Similar formulas can be obtained for the case $L_i = x_i$.