The p-center of the modular super Yangian $Y_{m|n}$

(Joint work with Hao Chang)

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Representation theory, Integrable Systems and Related Topics

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Drinfeld-type presentation of the modular super Yangian $Y_{m|n}$.

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Applications in classical representation theory

- J. Brundan, S.M. Goodwin, *A. Kleshchev, Highest weight theory for finite W-algebras*, Int. Math. Res. Not. **15** (2008).
- J. Brundan, A. Kleshchev, *Representations of shifted Yangians and finite W-algebras*, Mem. Am. Math. Soc. **196** (2008).
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- W. Wang, L. Zhao, *Representations of Lie superalgebras in prime characteristic I*, Proc. Lond. Math. Soc. **99** (2009) 145–167.
- Y. Zeng, B. Shu, *On Kac-Weisfeiler modules for general and special linear Lie superalgebras*, Isr. J. Math. **214** (2016) 471–490.

Restricted current superalgebra

- $\mathbb{k}$: an algebraically closed field with $\mathrm{char}(\mathbb{k}) =: p > 0$;
- $\mathfrak{g}$: the *current superalgebra* $\mathfrak{gl}_{m|n}[x] := \mathfrak{gl}_{m|n} \otimes \mathbb{k}[x]$;
  $\{e_{i,j}x^r := e_{i,j} \otimes x^r | r = 0, 1, 2, \dots, i, j = 1, \dots, m+n\}$.

$$\deg e_{i,j}x^r = |i| + |j|.$$

Restricted Lie superalgebra

$\mathfrak{g}$ is a restricted Lie superalgebra: for $a \in \mathfrak{gl}_{m|n}[x]_{\bar{0}}$

- the p-map defined on the basis: $(ax^r)^{[p]} := a^{[p]}x^{rp}$.

The center of $U(\mathfrak{g})$

$Z(\mathfrak{g})$ is freely generated by

- Harish-Chandra center:
  $\{z_r := e_{1,1}x^r + \cdots + e_{m+n,m+n}x^r;\ r \geq 0\}$;
- the p-center:
  $\{(e_{i,j}x^r)^p - \delta_{i,j}e_{i,j}x^{rp}; (i,j) \neq (1,1), r \geq 0,\ |i| + |j| = 0\}$.

RTT presentation of the modular Super Yangian $Y_{m|n}$

Super version

The super Yangian $Y_{m|n}$ associated to $\mathfrak{gl}_{m|n}$ over $\mathbb{C}$ was defined by Nazarov in 1991 (RTT-method).

Definition

The super Yangian $Y_{m|n}$, is the associated superalgebra over $\mathbb{k}$ with the generators $\{t_{i,j}^{(r)};\ 1 \leq i,j \leq m+n, r \geq 1\}$ subject to the following relations:

$$\left[t_{i,j}^{(r)}, t_{k,l}^{(s)}\right] = (-1)^{|i||j|+|i||k|+|j||k|} \sum_{t=0}^{\min(r,s)-1} \left(t_{k,j}^{(t)} t_{i,l}^{(r+s-1-t)} - t_{k,j}^{(r+s-1-t)} t_{i,l}^{(t)}\right)$$

- the parity of $t_{i,j}^{(r)}$ is defined by $|i| + |j| \pmod 2$, and $|i| = 0$ if $1 \leq i \leq m$ and $|i| = 1$ if $m < i \leq m+n$;
- Set $t_{i,j}^{(0)} := \delta_{i,j}$;
- The left side is super Lie-bracket.

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Equivalent defining relations

The formal power series: $t_{i,j}(u) := \sum_{r\geq 0} t_{i,j}^{(r)} u^{-r} \in Y_{m|n}[[u^{-1}]]$.

$$T(u) := \big(t_{i,j}(u)\big)_{1\leq i,j\leq m+n}.$$

Equivalent Expression

$$[t_{i,j}(u), t_{k,l}(v)] = \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{(u-v)}(t_{k,j}(u)t_{i,l}(v) - t_{k,j}(v)t_{i,l}(v)).$$

$$R(u-v)T_1(u)T_2(v) = T_2(v))T_1(u)R(u-v)$$

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$$R(u-v)T_1(u)T_2(v) = T_2(v))T_1(u)R(u-v)$$

Loop filtration

$\deg t_{i,j}^{(r)} = r-1$ is to define the *loop filtration* on $Y_{m|n}$ such that

$$Y_{m|n} = \bigcup_{r \geq 0} \mathrm{F}_r Y_{m|n}.$$

Lemma

The assignment $t_{i,j}^{(r)} \mapsto (-1)^{|i|} e_{i,j} x^{r-1}$ gives rise to the following isomorphism of graded superalgebras

$$\begin{array}{ccc} \mathrm{gr}\, Y_{m|n} & \cong & U(\mathfrak{g}) \\ ? & \leftrightarrow & \text{the center of } U(\mathfrak{g}) \end{array}$$

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Quasideterminant

Definition

Let A be an $n \times n$ matrix over a ring with 1. If the matrix A^{ij} is invertible, then the ij-th quasideterminant of A is defined by

$$|A|_{ij} := a_{ij} - r_i^j (A^{ij})^{-1} c_j^i = \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{n,n}. \end{vmatrix}.$$

Note that the leading minors of the generator matrix $T(u)$ of $Y_{m|n}$ are always invertible.

Gauss decomposition

The matrix $T(u)$ possesses a Gauss decomposition

$$T(u) = F(u)D(u)E(u)$$

for unique matrices

$$D(u) = \begin{pmatrix} d_1(u) & 0 & \cdots & 0 \\ 0 & d_2(u) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{m+n}(u) \end{pmatrix}; \quad E(u) = \begin{pmatrix} 1 & e_{1,2}(u) & \cdots & e_{1,m+n}(u) \\ 0 & 1 & \cdots & e_{2,m+n}(u) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix};$$

$$F(u) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f_{2,1}(u) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_{m+n,1}(u) & f_{m+n,2}(u) & \cdots & 1 \end{pmatrix}.$$

Continued.

$$d_i(u) = \begin{vmatrix} t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\ t_{i,1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{i,i}(u)} \end{vmatrix},$$

$$e_{i,j}(u) = d_i(u)^{-1} \begin{vmatrix} t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,j}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,j}(u) \\ t_{i,1}(u) & \cdots & t_{i,i-1}(u) & \boxed{t_{i,j}(u)} \end{vmatrix},$$

$$f_{j,i}(u) = \begin{vmatrix} t_{1,1}(u) & \cdots & t_{1,i-1}(u) & t_{1,i}(u) \\ \vdots & \ddots & \vdots & \vdots \\ t_{i-1,1}(u) & \cdots & t_{i-1,i-1}(u) & t_{i-1,i}(u) \\ t_{j,1}(u) & \cdots & t_{j,i-1}(u) & \boxed{t_{j,i}(u)} \end{vmatrix} d_i(u)^{-1}.$$

Drinfeld-type presentation

We use the following notation for the coefficients:

$$d_i(u) = \sum_{r\geq 0} d_i^{(r)} u^{-r}; \quad (d_i(u))^{-1} = \sum_{r\geq 0} d_i'^{(r)} u^{-r};$$

$$e_{i,j}(v) = \sum_{r\geq 1} e_{i,j}^{(r)} v^{-r}; \quad f_{j,i}(v) = \sum_{r\geq 1} f_{j,i}^{(r)} v^{-r}.$$

Let $e_j(u) := e_{j,j+1}(u)$, $f_j(v) := f_{j+1,j}(v)$ for short.

$$e_{i,j}^{(r)} = (-1)^{|j-1|}[e_{i,j-1}^{(r)}, e_{j-1}^{(1)}]; \ f_{j,i}^{(r)} = (-1)^{|j-1|}[f_{j-1}^{(1)}, f_{j-1,i}^{(r)}].$$

Theorem (Chang-Hu, J. Lond Math. Soc. 2023)

The Yangian $Y_{m|n}$ is generated by the elements $\{d_i^{(r)}, d_i'^{(r)}; \ 1 \leq i \leq m+n, r \geq 1\}$ and $\{e_j^{(r)}, f_j^{(r)}; \ 1 \leq j \leq m+n-1, r \geq 1\}$ subject only to the following relations:

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Drinfeld-type presentation

$$d_i^{(0)} = 1, \quad \sum_{t=0}^{r} d_i^{(t)} d_i'^{(r-t)} = \delta_{r0}; \quad [d_i^{(r)}, d_j^{(s)}] = 0;$$

$$[d_i^{(r)}, e_j^{(s)}] = (-1)^{|i|}(\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-1-t)}; \quad [d_i^{(r)}, f_j^{(s)}] = -(-1)^{|i|}(\delta_{ij} - \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-1-t)} d_i^{(t)};$$

$$[e_i^{(r)}, f_j^{(s)}] = -(-1)^{|i+1|} \delta_{ij} \sum_{t=0}^{r+s-1} d_i'^{(t)} d_{i+1}^{(r+s-1-t)}; \quad [e_i^{(r)}, e_j^{(s)}] = 0 = [f_i^{(r)}, f_j^{(s)}] \quad \text{if } |i-j| > 1;$$

$$[e_j^{(r+1)}, e_{j+1}^{(s)}] - [e_j^{(r)}, e_{j+1}^{(s+1)}] = (-1)^{|j+1|} e_j^{(r)} e_{j+1}^{(s)}; \quad [f_j^{(r+1)}, f_{j+1}^{(s)}] - [f_j^{(r)}, f_{j+1}^{(s+1)}] = -(-1)^{|j+1|} f_{j+1}^{(s)} f_j^{(r)};$$

$$[e_j^{(r)}, e_j^{(s)}] = (-1)^{|j+1|}\left(\sum_{t=1}^{s-1} e_j^{(t)} e_j^{r+s-1-t} - \sum_{t=1}^{r-1} e_j^{(t)} e_j^{(r+s-1-t)}\right);$$

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cubic Serre relations for $|i-j| = 1$:

$$\begin{cases} [[e_i^{(r)}, e_j^{(s)}], e_j^{(t)}] + [[e_i^{(r)}, e_j^{(t)}], e_j^{(s)}] = 0, \\ [[f_i^{(r)}, f_j^{(s)}], f_j^{(t)}] + [[f_i^{(r)}, f_j^{(t)}], f_j^{(s)}] = 0, \\ [[e_i^{(r)}, e_j^{(t)}], e_j^{(t)}] = 0; \quad [[f_i^{(r)}, f_j^{(t)}], f_j^{(t)}] = 0; \end{cases}$$

Super phenomenon: $[[e_{i-1}^{(r)}, e_i^{(1)}], [e_i^{(1)}, e_{i+1}^{(s)}]] = 0; \quad [[f_{i-1}^{(r)}, f_i^{(1)}], [f_i^{(1)}, f_{i+1}^{(s)}]] = 0.$

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The center of $Y_{m|n}$

- To describe Harish-Chandra center of $Y_{m|n}$

Thanks to *quantum Berezinian* (defined by Nazarov) of $T(u)$:

$$c(u) = d_1(u)d_2(u-1)\cdots d_m(u-m+1)\times d_{m+1}(u-m+1)^{-1}d_{m+n}\cdots(u-m+n)^{-1} =: 1 + \sum_{r\geq 1} c^{(r)}u^{-r}.$$

The elements $\{c^{(r)};\ r \geq 1\}$ are central. Furthermore, we have that $c^{(r)}$ has degree $r-1$ with respect to the loop filtration and $\mathrm{gr}_{r-1}c^{(r)} = z_{r-1} \in U(\mathfrak{g})$. Hence, $c^{(1)}, c^{(2)}, \ldots$ are algebraically independent.

- To describe the diagonal p-central elements of $Y_{m|n}$

$$b_i(u) := \begin{cases} d_i(u)d_i(u-1)\cdots d_i(u-p+1) & |i| = 0, \\ d_i(u)^{-1}d_i(u-1)^{-1}\cdots d_i(u-p+1)^{-1} & |i| = 1. \end{cases} := \sum_{r\geq 0} b_i^{(r)}u^{-r};$$

The elements $b_i^{(rp)}$ satisfy

$$\mathrm{gr}_{rp-p}\, b_i^{(rp)} = (e_{i,i}x^{r-1})^p - e_{i,i}x^{rp-p}.$$

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The center of $Y_{m|n}$

- To describe the off-diagonal p-central elements of $Y_{m|n}$

A consecutive application of the swap map and the anti-automorphism between the modular super Yangians, make us obtain that all coefficients $e_{i,j}^{(r)}$ and $f_{j,i}^{(r)}$ in the power series $(e_{i,j}(u))^p$ and $(f_{j,i}(u))^p$, respectively, belong to $Z(Y_{m|n})$. Then we define the *p-centre* $Z_p(Y_{m|n})$ of $Y_{m|n}$ to be the subalgebra generated by

$$\{b_i^{(rp)};\ 1 \le i \le m+n, r>0\} \cup \left\{(e_{i,j}^{(r)})^p, (f_{j,i}^{(r)})^p;\ 1 \le i < j \le m+n, r>0, |i|+|j|=0\right\}.$$

- To describe $Z_{\mathrm{HC}}(Y_{m|n}) \cap Z_p(Y_{m|n})$

$$\begin{aligned} bc(u) &:= b_1(u)b_2(u-1)\cdots b_m(u-m+1)b_{m+1}(u-m+1)\cdots b_{m+n}(u-m+n) \\ &= c(u)c(u-1)\cdots c(u-p+1) := \textstyle\sum_{r\ge 0} bc^{(r)}u^{-r}. \end{aligned}$$

then we prove that $bc^{(r)} \in Z_{\mathrm{HC}}(Y_{m|n}) \cap Z_p(Y_{m|n})$.

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The center of $Y_{m|n}$

Theorem (Chang-Hu, J. Lond Math. Soc. 2023)

The centre $Z(Y_{m|n})$ is generated by $Z_{\mathrm{HC}}(Y_{m|n})$ and $Z_p(Y_{m|n})$. Moreover:

1. $Z_{\mathrm{HC}}(Y_{m|n})$ is the free polynomial algebra generated by $\{c^{(r)};\ r > 0\}$;
2. $Z_p(Y_{m|n})$ is the free polynomial algebra generated by

$$\{b_i^{(rp)};\ 1 \leq i \leq m+n, r > 0\} \cup \{(e_{i,j}^{(r)})^p, (f_{j,i}^{(r)})^p;\ 1 \leq i < j \leq m+n, r > 0, |i| + |j| = 0\};$$

3. $Z(Y_{m|n})$ is the free polynomial algebra generated by

$$\{b_i^{(rp)}, c^{(r)};\ 2 \leq i \leq m+n, r > 0\} \cup \{(e_{i,j}^{(r)})^p, (f_{i,j}^{(s)})^p;\ 1 \leq i < j \leq m+n, r > 0, |i| + |j| = 0\};$$

4. $Z_{\mathrm{HC}}(Y_{m|n}) \cap Z_p(Y_{m|n})$ is the free polynomial algebra generated by $\{bc^{(rp)};\ r > 0\}$.

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The special super Yangian $SY_{m|n}$

We define the special super Yangian associated to the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ as the following subalgebra of $Y_{m|n}$:

$$SY_{m|n} := \{x \in Y_{m|n};\ \mu_f(x) = x \text{ for all } f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]\}$$

• For any power series $f(u) \in 1 + u^{-1}\mathbb{k}[[u^{-1}]]$, there is an automorphism $\mu_f : Y_{m|n} \to Y_{m|n}$ defined by

$$\mu_f(T(u)) = f(u)T(u).$$

• We let

$$h_i(u) = \sum_{r \geq 0} h_i^{(r)} u^{-r} := -(-1)^{|i|} d_{i+1}(u) d_i(u)^{-1}.$$

The special super Yangian $SY_{m|n}$: presentation

Theorem (Chang-Hu, J. Lond Math. Soc. 2023)

The algebra $SY_{m|n}$ is generated by the elements

$$\{h_i^{(r)}, e_i^{(r)}, f_i^{(r)};\ 1 \le i < m+n, r > 0\}$$

subject only to the following relations:

$$[h_i^{(r)}, h_j^{(s)}] = 0,\ [e_i^{(r)}, f_j^{(s)}] = (-1)^{|i|+|i+1|}\delta_{i,j} h_i^{(r+s-1)},$$

$$[h_i^{(r)}, e_j^{(s)}] = 0 = [h_i^{(r)}, f_j^{(s)}] \quad \text{if } |i-j| > 1,$$

$$[h_{i-1}^{(r+1)}, e_i^{(s)}] - [h_{i-1}^{(r)}, e_i^{(s+1)}] = (-1)^{|i|} h_{i-1}^{(r)} e_i^{(s)},$$

$$[h_{i-1}^{(r)}, f_i^{(s+1)}] - [h_{i-1}^{(r+1)}, f_i^{(s)}] = (-1)^{|i|} f_i^{(s)} h_{i-1}^{(r)},$$

$$[h_i^{(r+1)}, e_i^{(s)}] - [h_i^{(r)}, e_i^{(s+1)}] = \begin{cases} -(-1)^{i+1}\left(h_i^{(r)} e_i^{(s)} + e_i^{(s)} h_i^{(r)}\right), & \text{if } i \neq m, \\ 0 & \text{if } i = m, \end{cases}$$

$$[h_i^{(r)}, f_i^{(s+1)}] - [h_i^{(r+1)}, f_i^{(s)}] = \begin{cases} -(-1)^{i+1}\left(f_i^{(s)} h_i^{(s)} + h_i^{(r)} f_i^{(s)}\right), & \text{if } i \neq m, \\ 0 & \text{if } i = m, \end{cases}$$

$$[h_{i+1}^{(r+1)}, e_i^{(s)}] - [h_{i+1}^{(r)}, e_i^{(s+1)}] = (-1)^{|i+1|} e_i^{(s)} h_{i+1}^{(r)},$$

$$[h_{i+1}^{(r)}, f_i^{(s+1)}] - [h_{i+1}^{(r+1)}, f_i^{(s)}] = (-1)^{|i+1|} h_{i+1}^{(r)} f_i^{(s)},$$

The p-center of $SY_{m|n}$

- Define
  $a_i(u) = \sum_{r\geq 0} a_i^{(r)} u^{-r} := h_i(u)h_i(u-1)\cdots h_i(u-p+1)$,

Define the *p-center* $Z_p(SY_{m|n})$ to be the subalgebra generated by

$$\{a_i^{(rp)};\ 1 \leq i < m+n, r > 0\} \cup \left\{(e_{i,j}^{(r)})^p, (f_{j,i}^{(r)})^p;\ 1 \leq i < j \leq m+n, r > 0, |i|+|j| = 0\right\}.$$

$\mathfrak{g}'$: the current superalgebra associated to $\mathfrak{sl}_{m|n}$.

Theorem (Chang-Hu, J. Lond Math. Soc. 2023)

- $\mathrm{gr}\, Z_p(SY_{m|n}) = Z_p(\mathfrak{g}')$;
- If $p \nmid (m-n)$, then $Z_p(SY_{m|n}) = Z(SY_{m|n})$.

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Another presentation over char $\mathbb{k} \neq 2$

The new presentation of $Y(sl_n)$ given by Drinfeld in 1987 is associated to the Cartan matrix.

1. For $c \in \mathbb{k}$, there is an automorphism $\eta_c : Y_{m|n} \to Y_{m|n}$ defined from $\eta_c(t_{i,j}(u)) = t_{i,j}(u-c)$, i.e. $\eta_c(t_{i,j}^{(r)}) = \sum_{s=1}^{r} \binom{r-1}{r-s} c^{r-s} t_{i,j}^{(s)}$.
2. Define $h_i^{(r+s-1)} := -(-1)^{|i|} \sum_{t=0}^{r+s-1} d_i'^{(t)} d_{i+1}^{(r+s-1-t)}$.
3. Define $\kappa_{i,s}, \xi_{i,s}^{\pm}$ for $i = 1, \ldots, m+n-1$ and $s \geq 0$:

$$\kappa_i(u) = \sum_{s\geq 0} \kappa_{i,s} u^{-s-1} := (-1)^{|i|} + \eta_{(-1)^{|i|}(i-m)/2}(h_i(u)),$$

$$\xi_i^{+}(u) = \sum_{s\geq 0} \xi_{i,s}^{+} u^{-s-1} := \eta_{(-1)^{|i|}(i-m)/2}(e_i(u)),$$

$$\xi_i^{-}(u) = \sum_{s\geq 0} \xi_{i,s}^{-} u^{-s-1} := \eta_{(-1)^{|i|}(i-m)/2}(f_i(u)).$$

Another presentation of the modular super Yangian

Proposition (Chang-Hu, J. Lond Math. Soc. 2023)

The Yangian $SY_{m|n}$ is generated by the elements $\{\kappa_{i,s}, \xi^{\pm}_{i,s};\ 1 \le i < m+n, s \ge 0\}$ subject only to the following relations:

$$[\kappa_{i,r}, \kappa_{j,s}] = 0, \quad [\xi^{+}_{i,r}, \xi^{-}_{j,s}] = (-1)^{|i|+|i+1|}\delta_{i,j}\kappa_{i,r+s}, \quad [\kappa_{i,0}, \xi^{\pm}_{j,s}] = \pm(-1)^{|i|}a_{i,j}\xi^{\pm}_{j,s},$$

$$[\kappa_{i,r}, \xi^{\pm}_{j,s+1}] - [\kappa_{i,s+1}, \xi^{\pm}_{j,l}] = \pm\frac{a_{i,j}}{2}(\kappa_{i,r}\xi^{\pm}_{j,s} + \xi^{\pm}_{j,s}\kappa_{i,r}), \text{ for } i,j \text{ not both } m,$$

$$[\kappa_{m,r+1}, \xi^{\pm}_{m,s}] = 0, \qquad [\xi^{\pm}_{m,r}, \xi^{\pm}_{m,s}] = 0,\ [\xi^{\pm}_{i,r}, \xi^{\pm}_{j,s}] = 0,\ |i-j| > 1;$$

$$[\xi^{\pm}_{i,r}, \xi^{\pm}_{j,s+1}] - [\xi^{\pm}_{i,r+1}, \xi^{\pm}_{j,s}] = \pm\frac{a_{i,j}}{2}(\xi^{\pm}_{i,r}\xi^{\pm}_{j,s} + \xi^{\pm}_{j,s}\xi^{\pm}_{i,r}), \text{ for } i,j \text{ not both } m$$

$$\left[\xi^{\pm}_{i,r}, [\xi^{\pm}_{i,s}, \xi^{\pm}_{j,t}]\right] + \left[\xi^{\pm}_{i,s}, [\xi^{\pm}_{i,r}, \xi^{\pm}_{j,t}]\right] = 0 \text{ if } |i-j| = 1, \quad \left[[\xi^{\pm}_{m-1,r}, \xi^{\pm}_{m,0}], [\xi^{\pm}_{m,0}, \xi^{\pm}_{m+1,r}]\right] = 0.$$

Another description of the p-center of $Y_{m|n}$

For $|i| + |j| = 0$, we define

$$s_{i,j}(u) = \sum_{r \geq 0} s_{i,j}^{(r)} u^{-r} := \begin{cases} t_{i,j}(u) t_{i,j}(u-1) \cdots t_{i,j}(u-p+1) & \text{if } |i| = 0, \\ t'_{i,j}(u) t'_{i,j}(u-1) \cdots t'_{i,j}(u-p+1) & \text{if } |i| = 1. \end{cases}$$

Lemma

All of the elements $s_{i,j}^{(r)}$ belong to the p-center $Z_p(Y_{m|n})$.

Theorem (Chang-Hu, J. Lond Math. Soc. 2023)

The p-center $Z_p(Y_{m|n})$ is freely generated by $\{s_{i,j}^{(rp)};\ 1 \leq i, j \leq m+n, r > 0, |i| + |j| = 0\}$.

To do in the future

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Y. Peng, *Parabolic presentations of the super Yangian $Y(\mathfrak{gl}_{M|N})$*. Commun. Math. Phys. **307** (2011), 229–259.

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- Parabolic Presentation of the Modular super Yangian $Y_{m|n}$;
- Extend the works related to shifted (super)Yangian to positive characteristic;

Chang, Hao and Hu, Hongmei, A note on the center of the super Yangian $Y_{m|n}$. J. Algebra **633** (2023), 648–665.


THANK
YOU!