Vertex operators of affine quantum groups vs toroidal $\mathfrak{gl}_1$ algebra

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Quantum affine $\mathfrak{sl}_n$

- $\hat{I}_n$ is the cyclic quiver with n vertices $\{0, 1, 2, \ldots, n-1\}$
- $U_q(\widehat{\mathfrak{sl}}_n)$ is generated by K_i, E_i and F_i for $i \in \hat{I}$
- the relations are

$$K_i K_j = K_j K_i, \qquad K_i E_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j, \qquad K_i F_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} F_j,$$

$$[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$\sum_{k=0}^{b_{ij}} (-1)^k \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_q E_i^k E_j E_i^{b_{ij}-k} = 0, \qquad \sum_{k=0}^{b_{ij}} (-1)^k \begin{bmatrix} b_{ij} \\ k \end{bmatrix}_q F_i^k F_j F_i^{b_{ij}-k} = 0,$$

where $b_{ij} = 1 - a_{ij}$

- Drinfeld-Jimbo coproduct

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes 1 + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + 1 \otimes F_i,$$

Quantum affine $\mathfrak{sl}_n$: Drinfeld presentation/coproduct

- I is the linear quiver with $n-1$ vertices $\{1, 2, \ldots, n-1\}$
- $U_q(\widehat{\mathfrak{sl}}_n)$ is generated by $K_i^{\pm}[\pm m]$, $E_i[k]$ and $F_i[k]$ for $m \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}$, $i \in I$, and the central element $q^{\pm\frac{1}{2}c}$
- Consider currents

$$K_i^{\pm}(z) = \sum K_i^{\pm}[\pm m] z^{\mp m} \qquad X^{\pm}(z) = \sum X_i^{\pm}[k] z^{-k}$$

- some relations
- Drinfeld coproduct

$$\begin{aligned}\Delta(K_i^{\pm}(z)) =& K_i^{\pm}(c_2^{\pm\frac{1}{2}} z) \otimes K_i(c_1^{\mp\frac{1}{2}} z),\\ \Delta(E_i(z)) =& E_i(c_2 z) \otimes K_i^{-}(z) + 1 \otimes E_i(z),\\ \Delta(F_i(z)) =& F_i(z) \otimes 1 + K_i^{+}(z) \otimes F_i(c_1 z),\end{aligned}$$

Vertex operators for $U_q(\widehat{\mathfrak{sl}}_n)$

$F_0, F_1, \dots, F_{n-1}$ – integrable, level 1 representations of $U_q(\widehat{\mathfrak{sl}}_n)$. Intertwining operators

$$\Phi\colon F_{i+1} \to \mathbb{C}^n[u^{\pm 1}] \otimes F_i \qquad \Psi\colon F_{i+1} \to F_i \otimes \mathbb{C}^n[u^{\pm 1}] \tag{1}$$

$$\Phi^*\colon \mathbb{C}^n[u^{\pm 1}] \otimes F_i \to F_{i+1} \qquad \Psi^*\colon F_i \otimes \mathbb{C}^n[u^{\pm 1}] \to F_{i+1} \tag{2}$$

Denote

$$\Phi^*_\epsilon[k]w = \Phi^*(u^k v_\epsilon \otimes w) \qquad \Psi^*_\epsilon[k]w = \Psi^*(w \otimes u^k v_\epsilon) \tag{3}$$

for $\epsilon = 0, \dots n-1$ and $k \in \mathbb{Z}$. Denote

$$\Phi^*_\epsilon(z) = \sum_k \Phi^*_\epsilon[k] z^{-k} \qquad \Psi^*_\epsilon(z) = \sum_k \Psi^*_\epsilon[k] z^{-k} \tag{4}$$

Proposition

- *Vertex operators are exponentials of Heisenberg for Drinfeld coproduct*
- *Vertex operators are NOT necessarily exponentials of Heisenberg for Drinfeld-Jimbo coproduct*

Deformed W-algebras

Consider parameters

$$q_1 = q^{-1} d \qquad q_2 = q^2 \qquad q_3 = q^{-1} d^{-1} \tag{5}$$

Theorem (M. Bershtein, G)

The following formula gives an action of $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n)$ on F_i

$$T_1(z) = \sum u_i \Psi_i^*(dz) \Phi_i(z)$$

for both Drinfeld-Jimbo and Drindelf coproduct

- $\mathcal{W}_{q_1,q_2}(\mathfrak{sl}_n)$ is presented by currents $T_1(z), \dots, T_{n-1}(z)$
- T_k is obtained using vertex operators for $\Lambda^k(\mathbb{C}^n)$

Quantum toroidal $\mathfrak{gl}_1$

- $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ is a Hopf algebra, depends on parameters q_1 and q_2
- generated by $P_{a,b}$ for $(a,b) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ and central elements c and c'
- analogue of PBW-theorem holds for $P_{a,b}$, c, c'
- there are surjections $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1) \twoheadrightarrow \mathcal{W}_{q_1,q_2}(\mathfrak{gl}_n)$

Fock module

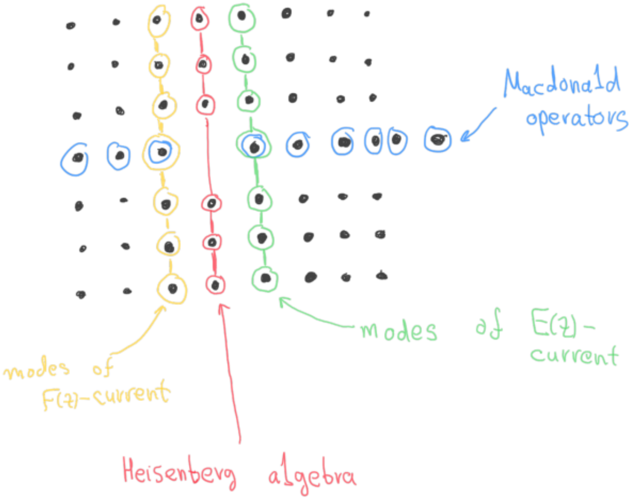
- Heisenberg algebra $[h_k, h_l] = k\delta_{k+l,0}$
- F is the Fock module for h_k

$U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ acts on F, the action is determined by

- $P_{0,k} \mapsto \# h_k$
- $P_{k,0}$ are Macdonald operators

Fock module

Consider formal power series of operators $E(z) = \sum_{k\in\mathbb{Z}} P_{1,k} z^{-k}$



Fock module and Chevalley generators

$P_{1,b}$, $P_{0,b}$ and $P_{-1,b}$ form another set of generators

Theorem (B. Feign, K. H Hashizume, A. Hoshino, J. Shiraishi, S. Yanagida)

The following formulas determine the action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$

$$c \mapsto q_2^{1/2} \qquad c' \mapsto 1 \qquad P_{0,b} \mapsto \# h_b \tag{6}$$

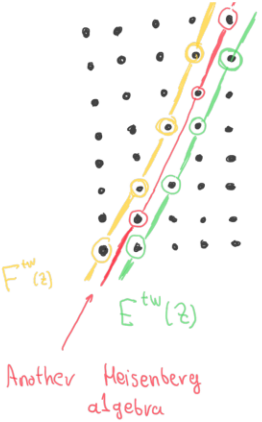
$$E(z) = \sum P_{1,b} z^{-b} \mapsto u\# :\exp\left(\sum_{k\neq 0} \# h_k z^{-k}\right): \tag{7}$$

$$F(z) = \sum P_{-1,b} z^{-b} \mapsto u^{-1}\# :\exp\left(\sum_{k\neq 0} \# h_k z^{-k}\right): \tag{8}$$

Denote the optained representation by $\mathcal{F}_u$

Twisted representation $\mathcal{F}^\sigma$

- $\widetilde{SL}_2(\mathbb{Z}) \curvearrowright U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$
- let M be a representation of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$ and $\sigma \in \widetilde{SL_2}(\mathbb{Z})$
- we get M^σ; we call it representation M *twisted* by σ.



The construction

Recall

$$q_1 = q^{-1} d \qquad q_2 = q^2 \qquad q_3 = q^{-1} d^{-1} \tag{9}$$

Let $\tilde{\Phi}_i(z)$, $\tilde{\Phi}_i^*(z)$, $\tilde{\Psi}_i(z)$, $\tilde{\Psi}_i^*(z)$ be $U_q(\widehat{\mathfrak{gl}})$ vertex operators for **Drinfeld-Jimbo** comultiplication

Theorem (M. Bershtein, G)

The formulas below define an action $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1) \curvearrowright F_i$

$$E^{tw}(z) = \sum_{a-b\equiv n'} \#\Psi_a^*(dz)\Phi_b(z) \tag{10}$$

$$F^{tw}(z) = \sum_{a-b\equiv -n'} \#\Phi_a^*(d^{-1}z)\Psi_b(z) \tag{11}$$

- for $\gcd(n', n) = 1$ we obtain twisted Fock
- $n' = 0$ corresponds to the tensor product of Fock modules

Deformed semi-infinite construction

- let $\hat{H}_N$ be the affine Hecke algebra for $\mathfrak{gl}_N$

$$U_q(\widehat{\mathfrak{gl}}_n) \curvearrowright \mathbb{C}^n[z_1^{\pm 1}] \otimes \cdots \otimes \mathbb{C}^n[z_N^{\pm 1}] \curvearrowleft \hat{H}_N$$

- let e_- be the deformed antisymmetrizer

$$\Lambda_q^N \mathbb{C}^n[z^{\pm 1}] = e_- \left(\mathbb{C}^n[z_1^{\pm 1}] \otimes \cdots \otimes \mathbb{C}^n[z_N^{\pm 1}]\right)$$

- the action of $\hat{H}_N$ can be extended to an action of double affine Hecke algebra $\mathcal{H}_N$
- we get from the above

$$U_q(\widehat{\mathfrak{gl}}_n) \curvearrowright \Lambda_q^N \mathbb{C}^n[z^{\pm 1}] \curvearrowleft e_- \mathcal{H}_N e_-$$

- taking the limit $N \to \infty$ ([KMS, LT] and [BG])

$$U_q(\widehat{\mathfrak{gl}}_n) \curvearrowright \Lambda_q^{\frac{\infty}{2}} \mathbb{C}^n[z^{\pm 1}] \curvearrowleft U_{q_1,q_2}(\ddot{\mathfrak{gl}}_1)$$

Toroidal $\mathfrak{gl}_n$

Let $\tilde{\Phi}_i(z)$, $\tilde{\Phi}_i^*(z)$, $\tilde{\Psi}_i(z)$, $\tilde{\Psi}_i^*(z)$ be $U_q(\widehat{\mathfrak{gl}}_n)$-vertex operators for Drinfeld compultiplication

Proposition (Ding, Iohara)

The following formulas determine the action of $U_q(\widehat{\mathfrak{gl}}_n)$

$$E_i(z) = \#\tilde{\Psi}_{i-1}(z)\tilde{\Psi}_i^*(z) \tag{12}$$

$$F_i(z) = \#\tilde{\Phi}_i(z)\tilde{\Phi}_{i-1}^*(z) \tag{13}$$

Proposition (G)

The action can be promoted to the action of $U_{q_1,q_2}(\ddot{\mathfrak{gl}}_n)$

$$E_0(z) = \#\tilde{\Psi}_{n-1}(d^n z)\tilde{\Psi}_0^*(z) \tag{14}$$

$$F_0(z) = \#\tilde{\Phi}_0(z)\tilde{\Phi}_{n-1}^*(d^n z) \tag{15}$$

Thank you for your attention!