Reduction by stages on W-algebras

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Motivations

History on W-algebras

Let Vir be the Virasoro algebra, which is an infinite-dimensional Lie algebra

$$\mathrm{Vir} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}C$$

with the defining relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}C, \quad [L_n, C] = 0.$$

The Virasoro algebra plays an important role in the 2d CFTs and also has a rich mathematical structure in the rep theory.

In the classification of 2d CFTs, Zamolodchikov found a generalization of Vir, called the W_3-algebra, which is generated by L_n, S_n, and L_n satisfies the Virasoro relations. However $[S_m, S_n]$ contains infinite sum of quadratic forms of L_n and thus W_3 is not just a Lie algebra – W_3 forms a vertex algebra.

Fateev and Lukyanov also found a family of generalizations of W_3-algebra: W_n-algebras (=WA_{n-1}), WB_n, WC_n, ... etc.

Feigin and Frenkel gave mathematical definitions of these algebras: let $\mathfrak{g}$ be a simple Lie algebra and $V^k(\mathfrak{g})$ the affine vertex algebra of $\mathfrak{g}$ at level k. Then the (principal) W-algebra of $\mathfrak{g}$ at level k is defined by the Drinfeld-Sokolov reduction

$$W^k(\mathfrak{g}) = H^0_{DS}(V^k(\mathfrak{g})).$$

By construction, the W-algebras are vertex algebras. For $\mathfrak{g} = \mathfrak{sl}_n$, $W^k(\mathfrak{sl}_n)$ is isomorphic to the W_n-algebra. In particular, $W^k(\mathfrak{sl}_2)$ is isomorphic to the Virasoro (vertex) algebra of the central charge $c(k) = 1 - 6(k+1)^2/(k+2)$.

Feigin and Semikhatov also found a family of genralizations of $V^k(\mathfrak{sl}_2)$, called the $W_n^{(2)}$-algebras. For $n = 2$, $W_2^{(2)} = V^k(\mathfrak{sl}_2)$ and $W_3^{(2)}$ is isomorphic to the Bershadsky-Polyakov algebra, which is geberated by e_n, h_n, f_n, L_n, and $[e_m, f_n]$ contains a linear term of L_n and infinite sum of quadratic forms of h_n, and thus not a Lie algebra. $W_n^{(2)}$ don't appear as examples of $W^k(\mathfrak{g})$.

Kac, Roan and Wakimoto found generalizations of $W^k(\mathfrak{g})$ by generalizing the DS-reductions: let $\mathfrak{g}$ be a simple Lie (super)algebra and f an (even) nilpotent element in $\mathfrak{g}$. Then

$$W^k(\mathfrak{g}, f) := H_f^0(V^k(\mathfrak{g})),$$

the (affine) W-algebra associated to $\mathfrak{g}, f$ at level k.

- $W^k(\mathfrak{g}, 0) = V^k(\mathfrak{g})$.
- $W^k(\mathfrak{g}, f_{\mathrm{prin}}) = W^k(\mathfrak{g})$, where f_{prin} is a principal nilp ele.
- $W^k(\mathfrak{sl}_n, f_{\mathrm{sub}}) = W_n^{(2)}$, where f_{sub} is a subregular nilp ele.

Madsen-Ragoucy observations

Consider the case $\mathfrak{g} = \mathfrak{sl}_3$. Let f be a nilpotent element in $\mathfrak{sl}_3$. Then the Jordan form of f has only 0 in the diagonal entries and thus is one of the followings:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which corresponds to the partitions (3), $(2,1)$, (1^3) of 3, called principal, subregular, zero, respectively. Thus we obtain two families of W-algebras from $V^k(\mathfrak{sl}_3)$:

- $W^k(\mathfrak{sl}_3, f_{\mathfrak{prin}}) =$ the Zamolodchikov W_3-algebra.
- $W^k(\mathfrak{sl}_3, f_{\mathfrak{sub}}) =$ the Bershadsky-Polyakov (BP) algebra.

Madsen and Ragoucy suggested the W_3-algebra is obtained from the BP-algebra by a quantum Hamiltonian reduction commuting the following diagram:

$$\begin{tikzcd}
V^k(\mathfrak{sl}_3) \arrow[rr, "H^0_{f_{\mathrm{prin}}}(?)"] \arrow[dr, "H^0_{f_{\mathrm{sub}}}(?)"'] & & \mathcal{W}^k(\mathfrak{sl}_3, f_{\mathrm{prin}}) \\
& \mathcal{W}^k(\mathfrak{sl}_3, f_{\mathrm{sub}}) \arrow[ur, dashed, "{}^{\exists}\text{quantum Hamiltonian reduction}"'] &
\end{tikzcd}$$

Questions:

1) Want to understand the reason why this happens.

2) Want to get generalizations.

Our goal is to prove the Madsen-Ragoucy observations and find generalizations using reduction by stages.

Family of W-algebras

Let V be a vertex algebra and $\mathrm{Zhu}\, V = V/(V \circ V)$, where

$$V \circ V = \mathrm{Span}\{a \circ b \mid a, b \in V\}, \quad a \circ b = \sum_{j=0}^{\infty} \binom{\Delta(a)}{j} a_{(j-2)} b,$$

$a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$ (vertex operators) and $\Delta(a)$ = the conformal weight of a. Then $\mathrm{Zhu}\, V$ becomes an associative algebra (Zhu algebra) by the product induced from

$$a * b = \sum_{j=0}^{\infty} \binom{\Delta(a)}{j} a_{(j-1)} b$$

and there exists one-to-one correspondence between simple modules of V and simple modules of $\mathrm{Zhu}\, V$ (the Zhu theorem).

· $V = V^k(\mathfrak{g}) \Rightarrow \text{Zhu}\, V^k(\mathfrak{g}) = U(\mathfrak{g})$ (I.Frenkel-Zhu).

· $V = W^k(\mathfrak{g}, f) \Rightarrow \text{Zhu}\, W^k(\mathfrak{g}, f) = U(\mathfrak{g}, f)$
(Arakawa, DeSole-Kac),

where $U(\mathfrak{g}, f)$ is the finite W-algebra of $\mathfrak{g}, f$, introduced by Premet (cf. Weinan & Hongmei's talk) defined by

$$U(\mathfrak{g}, f) = H^0_f(U(\mathfrak{g})).$$

· $U(\mathfrak{g}, 0) = U(\mathfrak{g})$

· $U(\mathfrak{g}, f_{\text{prin}}) = Z(\mathfrak{g})$: center of $U(\mathfrak{g})$ (Kostant)

· $U(\mathfrak{sl}_n, (m^\ell))$ = Cherednik's Yangian of level ℓ (Ragoucy-Sorba)

· $U(\mathfrak{sl}_n, f)$ = shifted Yangian of type A (Brundan-Klechshev)

· $U(\mathfrak{g}, (m^\ell))$ = shifted twisted Yangian of type BCD
if $\mathfrak{g} = BCD$ (Brown)

· $U(\mathfrak{sl}(m|n), f)$ = shifted super Yangian of type A
(Briot-Ragoucy, Brown-Brundan-Goodwin, Peng)

Recall: using the PBW filtration on $U(\mathfrak{g})$,

$$\mathrm{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \quad \text{(Possion algebra)}$$

Thus $\mathfrak{g}^*$ is a Poisson variety, and the symplectic leaves of $\mathfrak{g}^*$ are coadjoint orbits $\mathcal{O}^*$.

The finite W-algebra has a canonical filtration (Khazdan filtration), and the associated graded algebra also becomes a Poisson algebra (Premet, Gan-Ginzburg, Losev):

$$\mathrm{gr}\, U(\mathfrak{g}, f) \simeq \mathbb{C}[\mathcal{S}_f],$$

where $\mathcal{S}_f$ is the Slodowy slice of $\mathfrak{g}$ at f (defined in next slide). We have

$$\mathbb{C}[\mathcal{S}_f] = H_f^0(\mathbb{C}[\mathfrak{g}^*]).$$

Suppose that $f \neq 0$. Then the Jacobson-Morozov theorem implies that there exists an $\mathfrak{sl}_2$-triple $\{e, h, f\} \subset \mathfrak{g}$ containing our choice of f. Then

$$\mathcal{S}_f = f + \mathfrak{g}^e \subset \mathfrak{g} \simeq \mathfrak{g}^*.$$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $h \in \mathfrak{h}$.

A pair (f, h) is called a good pair if

(1) $\operatorname{ad} h$ defines a $\mathbb{Z}$-grading on $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j$, where

$\mathfrak{g}_j = \{a \in \mathfrak{g} \mid [h, a] = ja\}$

(2) $f \in \mathfrak{g}_{-2}$

(3) $\operatorname{ad} f \colon \mathfrak{g}_j \to \mathfrak{g}_{j-2}$ is injective for $j \geq 1$ and surjective for $j \leq 1$.

For example, we may choose h in the $\mathfrak{sl}_2$-triple $\{e, h, f\}$

(But in general, we have more options for h. Classifications: Elashvili-Kac, Hoyt).

By the good conditions,

$$\langle a, b \rangle := (f|[a, b]) = ([f, a]|b), \quad a, b \in \mathfrak{g}_1$$

defines a non-deg. skew-symmetric (=symplectic) form on $\mathfrak{g}_1$.

Let $\mathfrak{l}$ be a Lagrangian in $\mathfrak{g}_1$ (= a maximal isotropic subspace in $\mathfrak{g}_1$) and $\mathfrak{m}$ a nilpotent subalgebra

$$\mathfrak{m} = \mathfrak{l} \oplus \mathfrak{g}_{\geq 2}.$$

For example, in case $\mathfrak{g} = \mathfrak{sl}_3$,

$$f = f_{\mathrm{prin}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \mathfrak{m} = \mathfrak{n}_+ = \begin{pmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix},$$

$$f = f_{\mathrm{sub}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathfrak{m} = \begin{pmatrix} 0 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $M = \exp(\mathfrak{m})$ a unipotent Lie group. Then the coadjoint action of M on $\mathfrak{g}^*$ is Hamiltonian with the moment map

$$\mu\colon \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{m}^*, \quad a \mapsto (a\,|\,\cdot).$$

Let $\chi = (f\,|\cdot) \in \mathfrak{m}^*$. Gan-Ginzburg shows that

$$\mathcal{S}_f \simeq \mu^{-1}(\chi)/M =: \mathfrak{g}^*/\!/M.$$

The RHS is called the Hamiltonian reduction of $\mathfrak{g}^*$ by M at χ (Ivan Sechin's talk). Then the Poisson structure of $\mathcal{S}_f$ is induced from $\mathfrak{g}^*$.

Reduction by stages

Let X be a Poisson variety with a Hamiltonian M_2-action and M_1 a normal Lie subgroup of M_2. Then we obtain two Poisson varieties $X/\!/M_1$, $X/\!/M_2$ from X by using the Hamiltonian reductions. But, under suitable assumptions, we may define a Hamiltonian M_2/M_1-action on $X/\!/M_1$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "/\!/M_2"] \arrow[dr, "/\!/M_1"'] & & X/\!/M_2 \\ & X/\!/M_1 \arrow[ur, dashed, "/\!/(M_2/M_1)"'] & \end{tikzcd}$$

This procedure is called the reduction by stages since we obtain $X/\!/M_2$ by stages. We will apply for $X = \mathfrak{g}^*$.

Morgan conjectures

Morgan applied the reduction by stages for the Slodowy slices.

Conjecture (Morgan, PhD thesis)

Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathcal{O}_{f_1}, \mathcal{O}_{f_2}$ nilpotent orbits in $\mathfrak{sl}_n$ at f_1, f_2 such that $\mathcal{O}_{f_1} < \mathcal{O}_{f_2}$ (i.e. $\mathcal{O}_{f_1} \subset \overline{\mathcal{O}}_{f_2}$). Then $\mathcal{S}_{f_2}$ is obtained as a Hamiltonian reduction of $\mathcal{S}_{f_1}$.

Theorem (Morgan)

This is true for $n = 3$.

$$\begin{tikzcd}
\mathfrak{g}^* = \mathfrak{sl}_3^* \arrow[rr] \arrow[dr] & & \mathcal{S}_{f_{\mathrm{prin}}} = \mathfrak{g}^* /\!/ M_2 \\
& \mathcal{S}_{f_{\mathrm{sub}}} = \mathfrak{g}^* /\!/ M_1 \arrow[ur, dashed, "\exists \text{ Hamiltonian reduction}"'] &
\end{tikzcd}$$

This is a classical analog of the Madsen-Ragoucy observations.

Reduction by stages for Slodowy slices / finite W-algebras

Step conditions

Let (f_1, h_1), (f_2, h_2) be good pairs in $\mathfrak{g}$ s.t. $h_1, h_2 \in \mathfrak{h}$, and

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j^{(1)} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_j^{(2)}$$

the $\mathbb{Z}$-gradings by $\mathrm{ad}\, h_1$, $\mathrm{ad}\, h_2$. Then we have $\mathfrak{m}_1, \mathfrak{m}_2$, and $\mathcal{S}_{f_1} \simeq \mu_1^{-1}(\chi_1)/M_1$ and $\mathcal{S}_{f_2} \simeq \mu_2^{-1}(\chi_2)/M_2$.

Definition

(f_1, h_1) is a step towards (f_2, h_2) if $f_0 := f_2 - f_1 \in \mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{-2}^{(2)}$, and

$$\mathfrak{g}_{\geq 2}^{(1)} \subset \mathfrak{g}_{\geq 1}^{(2)} \subset \mathfrak{g}_{\geq 0}^{(1)}, \quad \mathfrak{g}_1^{(1)} \subset \bigoplus_{j=0}^{2} \mathfrak{g}_j^{(2)}, \quad \mathfrak{g}_1^{(2)} \subset \bigoplus_{j=0}^{2} \mathfrak{g}_j^{(1)}.$$

Note: we will explain examples later.

Main results for Slodowy slices / finite W-algebras

By the step conditions,

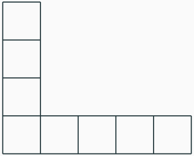
- the nilpotent orbits $\mathcal{O}_{f_1}$ of f_1 is contained in the Zariski closure $\overline{\mathcal{O}}_{f_2}$ of the nilpotent orbits of f_2 (i.e. $\mathcal{O}_{f_1} < \mathcal{O}_{f_2}$).
- $\mathfrak{m}_1 \subset \mathfrak{m}_2$ ideal and $\mathfrak{m}_2 = \mathfrak{m}_1 \oplus \mathfrak{m}_0$ with $\mathfrak{m}_0 := \mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{\geq 2}^{(2)}$.
- $\mathcal{S}_{f_1}$ has a Hamiltonian M_0-action, where $M_0 = \exp(\mathfrak{m}_0) \simeq M_2/M_1$.

Theorem (G.-Juillard)
Assume that (f_1, h_1) *is a step towards* (f_2, h_2)*. Then*

1. $\mathcal{S}_{f_2} \simeq \mathcal{S}_{f_1}//M_0$.
2. $U(\mathfrak{g}, f_2) \simeq H^0_{f_0}(U(\mathfrak{g}, f_1))$.

Examples:

- Let $\mathfrak{g} = \mathfrak{sl}_n$, $a_1, a_2 \in \mathbb{N}$ such that $1 \leq a_1 < a_2 \leq n$ and $f_1 = (a_1, 1^{n-a_1})$, $f_2 = (a_2, 1^{n-a_2})$. These are called hook-type nilpotent elements:



  Then f_1, f_2 satisfies the step conditions.
- Let $\mathfrak{g} = \mathfrak{sl}_4$, $f_1 = (2, 1^2)$ and $f_2 = (2^2)$.
- Let $\mathfrak{g} = \mathfrak{so}_{2n+1}$, f_1 is subregular and f_2 is principal.
- Let $\mathfrak{g} = \mathfrak{sp}_{2n}$, $f_1 = (2^2, 1^{2n-4})$ (short nilpotent) and f_2 is principal.
- Let $\mathfrak{g} = G_2$, f_1 is $\widetilde{A}_1$ and f_2 is subregular.
- (Maybe) more...

Applications: Skryabin equivalence by stages

Recall $\chi(u) = (f|u)$. Let $I_\chi = (u - \chi(u) \mid u \in \mathfrak{m})$ be a two-sided ideal in $U(\mathfrak{g})$ and $Q_\chi = U(\mathfrak{g})/I_\chi$. Then $Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$ has a structure of an associative algebra induced from $U(\mathfrak{g})$ and

$$U(\mathfrak{g}, f) \simeq Q_\chi^{\mathrm{ad}\,\mathfrak{m}}$$

(D'Andrea-DeConcini-DeSole-Heluani-Kac). The RHS is the original definition of Premet.

A $\mathfrak{g}$-module E is called Whittaker for χ if $u - \chi(u)$ acts on E locally nilpotently for all $u \in \mathfrak{m}$.

Let $\mathfrak{g}-\mathrm{mod}_\chi$ be the category of fin. gen. Whittaker $\mathfrak{g}$-modules for χ. For $E \in \mathfrak{g}-\mathrm{mod}_\chi$,

$$\mathrm{Wh}(E) := \{m \in E \mid (u - \chi(u))m = 0, u \in \mathrm{m}\}$$

becomes a $U(\mathfrak{g}, f)$-module.

Let $U(\mathfrak{g}, f)-\mathrm{mod}$ be the category of fin. gen. $U(\mathfrak{g}, f)$-modules. For $V \in U(\mathfrak{g}, f)-\mathrm{mod}$,

$$\mathrm{Ind}(V) := Q_\chi \bigotimes_{U(\mathfrak{g},f)} V$$

becomes a Whittaker $\mathfrak{g}$-module. Moreover,

$$\mathfrak{g}-\mathrm{mod}_\chi \underset{\mathrm{Ind}}{\overset{\mathrm{Wh}}{\rightleftarrows}} U(\mathfrak{g}, f)-\mathrm{mod}$$

gives a quasi-inverse category equivalence. This is called the Skryabin equivalence.

Theorem (G.-Juillard)
Assume that (f_1, h_1) is a step towards (f_2, h_2). Then the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{g}-\mathrm{mod}_{\chi_2} \arrow[rr, shift left, "\mathsf{Wh}_2"] \arrow[dr, shift left, "\mathsf{Wh}_1"] & & U(\mathfrak{g}, f_2)-\mathrm{mod} \arrow[ll, shift left, "\mathsf{Ind}_2"] \arrow[dl, shift left, "\mathsf{Ind}_0"] \\
& U(\mathfrak{g}, f_1)-\mathrm{mod}_{\chi_0} \arrow[ul, shift left, "\mathsf{Ind}_1"] \arrow[ur, shift left, "\mathsf{Wh}_0"] &
\end{tikzcd}$$

and each $\rightleftarrows$ are quasi-inverse equivalences.

Remark: E. Masut shows that Theorem is compatible with Goodwin's translation functors (arXiv:2404.07859).

Reduction by stages for affine W-algebras

Twisted Gan-Ginzburg Theorem

Recall: $W^k(\mathfrak{g}, f) = H^0_f(V^k(\mathfrak{g}))$.
Claim: $W^k(\mathfrak{g}, f_2) \simeq H^0_{f_0}(W^k(\mathfrak{g}, f_1))$.
Want:

$$H^0_{f_2}(V^k(\mathfrak{g})) \dashrightarrow H^0_{f_0}(H^0_{f_1}(V^k(\mathfrak{g}))),$$

but it is difficult to find such a good map (technical difficulty).

By the step conditions,

$\cdot\ \mathfrak{n}_1 := \mathfrak{g}^{(1)}_{\geq 1} \supset \mathfrak{m}_1, \quad \cdot\ \mathfrak{n}_2 := \mathfrak{n}_1 \oplus \mathfrak{m}_0 \supset \mathfrak{m}_2$.

Then $N_i = \exp(\mathfrak{n}_i)$ acts on $\mathfrak{n}_i^*$ by the coadjoint action ($i = 1, 2$).
Set $\mathcal{O}_i = N_i \cdot \chi_i \subset \mathfrak{n}_i^*$ with $\chi_i = (f_i \mid \cdot) \in \mathfrak{n}_i^*$.

Proposition (twisted version of Gan-Ginzburg)

$\mathcal{S}_{f_i} \simeq \mu_i^{-1}(\mathcal{O}_i)/N_i$.

$\Rightarrow \mathbb{C}[\mathcal{S}_{f_i}] \simeq H^0_{\mathcal{O}_i}(\mathbb{C}[\mathfrak{g}^*])$.

Proposition

$U(\mathfrak{g}, f_i) \simeq H^0_{\mathcal{O}_i}(U(\mathfrak{g}))$.

Hence

$$H^0_{\mathcal{O}_2}(U(\mathfrak{g})) \simeq H^0_{f_0}(H^0_{\mathcal{O}_1}(U(\mathfrak{g}))).$$

This is, in fact, the double complex isomorphism.

Claim

$W^k(\mathfrak{g}, f_i) \simeq H^0_{\mathcal{O}_i}(V^k(\mathfrak{g}))$.

Then we can find a vertex algebra homomorphism

$$H^0_{\mathcal{O}_2}(V^k(\mathfrak{g})) \to H^0_{f_0}(H^0_{\mathcal{O}_1}(V^k(\mathfrak{g}))).$$

Main results for affine W-algebras

Conjecture

$\operatorname{gr} H^0_{\mathcal{O}_i}(V^k(\mathfrak{g})) \simeq \mathbb{C}[J_\infty \mathcal{S}_{f_i}]$.

- In the LHS, gr is defined by the Li filtration.
- In the RHS, $J_\infty \mathcal{S}_{f_i}$ = the arc space of $\mathcal{S}_{f_i}$
- If the conjecture is true, the claim holds.
- The conjecture is true for hook-types f_1, f_2 in $\mathfrak{sl}_n$.

Theorem (G.-Juillard, in progress)

Assume that (f_1, h_1) *and* (f_2, h_2) *satisfy the step conditions and Conjecture. Then*

$$W^k(\mathfrak{g}, f_2) \simeq H^0_{f_0}(W^k(\mathfrak{g}, f_1)).$$