Kac-Moody algebras and beyond via DAHA

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Almost by design, DAHA serve refined theories (with q, t, a), toroidal and elliptic algebras. The passage to Kac-Moody algebras is generally for $t \to 0, \infty$ or via Verlinde algebras, which can be nonsymmetric and non-semisimple in the DAHA theory. The usual ones are for $t = q$, $q^N = 1$ and upon the symmetrization. Examples of nonsymmetric applications of DAHA: level-1 Demazure characters and boundary-level Kac-Wakimoto representations. We will mostly focus on refined Rogers-Ramanujan series; they are governed by $2d$ TQFT with levels. They satisfy the superduality, a recent theorem, and can be viewed as invariants of some lens spaces.

Let $R=\{\alpha\}\in\mathbb{R}^n$ be a simple root system, $(\cdot,\cdot)$ the corresponding inner product normalized by $(\boldsymbol{\alpha}_{\rm sht},\boldsymbol{\alpha}_{\rm sht})=\mathbf{2}$, $\{\alpha_i\}$ simple roots, $W=\langle s_i=s_{\alpha_i}\rangle=\langle s_\alpha\rangle$ the Weyl group, $\rho_k=(1/2)\sum_{\alpha>0}k_\alpha\alpha$, $P=\oplus_i\mathbb{Z}\omega_i$ the weight lattice (for fundamental ω_i), $P_+=\oplus\mathbb{Z}_+\omega_i$, $Q=\sum_\alpha\mathbb{Z}\alpha, Q_+=\sum_{\alpha>0}\mathbb{Z}_+\alpha$. We set $\mathbb{C}[X_a]=\mathbb{C}[X_{\omega_i}^{\pm1}]$, where $X_{a+b}=X_aX_b$ for $a,b\in P$, $w(X_a)=X_{w(a)}$ for $w\in W$, $\mathbb{C}[X]^W=\{F\in\mathbb{C}[X_a], w(F)=F\}$, $\langle \boldsymbol{F}\rangle$ the constant term of Laurent series $\boldsymbol{F}$, $X_a^\iota=X_{\iota(a)}$, where $\iota(a)=-w_0(a)$ for the longest element $w_0\in W$.

Let $\theta_u(X)\stackrel{\mathbf{def}}{=\!=}\sum_{a\in P}u(a)q^{(a,a)/2}X_a, \theta=\theta_{triv}$ for characters $u: P/Q\to\mathbb{C}^*$, playing the role of the classical theta- characteristics (necessary in the level-rank duality for R of type A). Also: $\theta_{\mathbf{u}}^{(\ell)}=\theta_{u_1}\cdots\theta_{u_\ell}$ for $\mathbf{u}=\{u_1,\ldots,u_\ell\},\ell\geq 0$. We will focus on:

Littlewood-Richardson formulas. Given a system of orthogonal polynomials $\{P_a, a\in P_+\}$ linearly generating $\mathbb{C}[X]^W$, the problem is to calculate/interpret $\widetilde{P}_a\widetilde{P}_b=\sum_c\mathbb{C}_{ab}^{c\mathbf{u}}\widetilde{P}_c$ for $\widetilde{P}_a\stackrel{\mathbf{def}}{=\!=}P_a\theta_{\mathbf{u}}^{(\ell)}, a,b\in P_+$.

DAHA: TWO APPROACHES

For $E = T^2$, we set $\mathcal{H\!H} = \mathbb{C}\mathbf{B}_{ell}/\{T_i^2 + aT_i + b = 0\}$ for $\mathbf{B}_{ell} = \pi_1\big(\big(E^n \setminus \{x \mid \prod_\alpha x_\alpha = 0\}\big)/W\big)$; $T_i (1 \le i < n)$ are the usual "half-turns" for any irreducible reduced root system $R \in \mathbb{R}^n$; the orbifold π_1 is used. In this approach, the action of the projective $PSL_2(\mathbb{Z})$ ($= B_3$ due to Steinberg) in $\mathcal{H\!H}$ is granted, which is far from obvious via $K_{T\times C^*}(\widehat{G/B})$, the 2nd major general approach. Algebraically, DAHA is a universal flat deformation of the Heisenberg-Weyl algebra extended by W. Its Fock representation is the polynomial representation $\mathcal{X}$. The eigenfunctions of "Y-operators" are nonsymmetric Macdonald polynomials. The symmetric polynomials are obtained upon the t-symmetrization. The limit $t \to 0$ results in nil-DAHA and generalized Hermite polynomials.

A_1-DAHA, PROJECTIVE $PSL_2(Z)$-ACTION

For A_1, $\mathcal{H} \stackrel{\mathbf{def}}{=\!=} \langle T, X^{\pm 1}, Y^{\pm 1}, t^{\pm\frac{1}{2}}, q^{\pm\frac{1}{4}} \rangle$

subject to relations: $TXTX = 1 = TY^{-1}TY^{-1}$,

$Y^{-1}X^{-1}YXT^2 = q^{-1/2}$, $(T - t^{\frac{1}{2}})(T + t^{-\frac{1}{2}}) = 0$;

$\widetilde{PSL_2}(\mathbb{Z}) \ni \tau_\pm$, $\tau_+ : Y \mapsto q^{-\frac{1}{4}} XY, X \mapsto X, T \mapsto T$.

For $t = 1$: $\mathcal{H}$ = (Weyl algebra)$\rtimes$S$_2$ setting $T \to s$.

$$\mathcal{H} \circlearrowright \mathcal{X} = \mathbb{C}[X^{\pm 1}] : \; T \mapsto t^{1/2}s + \frac{t^{1/2} - t^{-1/2}}{X^2 - 1}(s - 1),$$

$$X \mapsto X, \; Y \mapsto \pi T, \, \pi = sp, \; s(X) = X^{-1}, \; p(X) = q^{1/2}X.$$

For GL_n, $\tau_+(Y_1) = q^{-1/2}X_1Y_1$, $\tau_-(X_1) = q^{+1/2}Y_1X_1$,
$Y_1 = \pi T_{n-1} \ldots T_1$, $\pi : X_1 \mapsto X_2, \ldots, X_n \mapsto q^{-1}X_1, \ldots$

REFINED VERLINDE ALGEBRAS

Let $q = \exp(\frac{2\pi i}{N})$, $k < N/2, k \in \frac{\mathbb{Z}_+}{2}$. The map $X(z) = q^z$ can be extended to an $\mathcal{H\!H}$-homomorphism $\mathbb{C}[X^{\pm 1}] \to V \stackrel{\text{def}}{=\!=} Funct\{-\frac{N+k+1}{2}, \ldots, -\frac{k+1}{2}, -\frac{k}{2}, \frac{k+1}{2}, \ldots, \frac{N-k}{2}\}$. This is Non-symmetric Verlinde Algebra. Generally, they are perfect representations, which are canonical irreducible finite-dimensional quotients of the polynomial representation $\mathcal{X}$ of $\mathcal{H\!H}$ if they exist (at roots of unity q or if $t = q^k$ for some fractional k).

In V above, X, Y, T are unitary in V for the "minimal" primitive Nth root q. Also, $PSL_2(\mathbb{Z})$ acts in V projectively and in the image $V_{sym} = \{f \in V \mid Tf = t^{\frac{1}{2}} f\}$ of $\mathbb{C}[X^{\pm 1}]_{sym}$, which, generally, follows from their rigidity. This is a far-reaching generalization (nonsymmetric and, possibly, non-semisimple) of the action of $PSL_2(\mathbb{Z})$ on the Kac-Moody characters.

QUANTUM GROUPS AT ROOTS OF UNITY

Thus, $\dim_{\mathbb{C}} V = 2N-4k$, $\dim_{\mathbb{C}} V_{sym} = N-2k+1$. **Usual Verlinde algebra is Ver** $= V_{sym}^{k=1}$; τ_+ **becomes the** T**-operator,** $\sigma = \tau_+\tau_-^{-1}\tau_+$ **, which is generally DAHA Fourier transform , becomes the** S**-operator. Generally, the "characters" in Verlinde algebras are replaced by eigenfunctions of** Y **and** $Y+Y^{-1}$ **in** V **and** V_{sym}**, the images of the Macdonald polynomials.**

Ver represents integral irreducible Kac-Moody modules of level $N-h+1$ **with fusion. Conjecturally, the spherical part of the whole polynomial representation when** $t = q$ **(equal parameters) and** q **is a root of unity is** Rep_q **of Lusztig's quantum group; Ver =reduced category. Also, its quotient by the canonical central character, describes** Rep_q **of of small quantum group. Then** $PSL_2(\mathbb{Z})$ **acts in irreducible (spherical) DAHA constituents, but NOT in the whole** Rep_q**.**

ROGERS-RAMANUJAN SERIES

Let c_+ **be such that** $c_+ \in W(c) \cap P_+$**. Given** $b \in P_+$**, let** $b \neq c_+ \in b - Q_+$**,** $P_b - \sum_{a\in W(b)} X_a \in \oplus_c \mathbb{C} X_c$**,** $\langle\, P_b X_{c^\iota}\, \mu(X;q,t)\,\rangle = 0$ **for such** c**, where** $\mu(X;q,t) \stackrel{\mathbf{def}}{=\!=} \prod_{\alpha\in R_+} \prod_{j=0}^{\infty} \frac{(1-X_\alpha q_\alpha^j)(1-X_\alpha^{-1} q_\alpha^{j+1})}{(1-X_\alpha t_\alpha q_\alpha^j)(1-X_\alpha^{-1} t_\alpha q_\alpha^{j+1})}$ **, considered a Laurent series of** X_b **(expanded in terms of positive powers of** q**),** $q_\alpha = q^{\nu_\alpha}, \nu_\alpha = \frac{(\alpha,\alpha)}{2}, t_\alpha = t_{\nu_\alpha}$**; the coefficients of** P_b **belong to the field** $\mathbb{Q}(q, t_\nu)$**. Setting** $t_\alpha = q_\alpha^{k_\alpha}$**,** $k_\alpha = k_{\nu_\alpha}$**,** $X_a(q^b) = q^{(a,b)}$**,** $P_b(q^{\rho_k}) = q^{-(\rho_k,b)} \prod_{\alpha>0} \prod_{j=0}^{(\alpha^\vee,b)-1} \Big(\frac{1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k})}{1-q_\alpha^j X_\alpha(q^{\rho_k})}\Big)$**,** $\langle P_b P_c^\iota \mu\rangle = \langle\mu\rangle\, \delta_{bc} \prod_{\alpha>0} \prod_{j=0}^{(\alpha^\vee,b)-1} \frac{(1-q_\alpha^{j+1} t_\alpha^{-1} X_\alpha(q^{\rho_k}))(1-q_\alpha^j t_\alpha X_\alpha(q^{\rho_k}))}{(1-q_\alpha^j X_\alpha(q^{\rho_k}))(1-q_\alpha^{j+1} X_\alpha(q^{\rho_k}))}$.

For any $b, c \in P_+$**,** $\mathbf{u} = (u_1, \ldots, u_\ell)$**, and** $\mathbb{C}_{ab}^{c\mathbf{u}}$ **for** $\theta_{\mathbf{u}}^{(\ell)}$ **above:**

$\mathbb{C}_{0b}^{cu} \stackrel{\mathbf{def}}{=\!=} \frac{\langle P_b P_c^\iota \theta_u\, \mu\rangle}{\langle P_c P_c^\iota \mu\rangle} = \frac{q^{b^2/2+c^2/2+(b+c,\rho_k)}}{u(b-c)\langle P_c P_c^\iota \mu\rangle} P_b^\iota(q^{c+\rho_k}) P_c(q^{\rho_k})\langle\theta\mu\rangle,$

$\mathbb{C}_{\mathbf{0b}}^{\mathbf{cu}} = \sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \mathbb{C}_{\mathbf{0b}}^{\mathbf{c_1 u_1}} \mathbb{C}_{\mathbf{0c_1}}^{\mathbf{c_2 u_2}} \mathbb{C}_{\mathbf{0c_2}}^{\mathbf{c_3 u_3}} \cdots \mathbb{C}_{\mathbf{0c_{\ell-1}}}^{\mathbf{c u_\ell}}$. **(R-R)**

STABILIZATION AND SUPERDUALITY

The series for $\mathbb{C}^{0\mathbf{u}}_{00}$ are refined Rogers-Ramanujan sums; they become modular 0-weight functions as $t \to 0$ [*Ch,B.Feigin,2013*]. For $\mathbf{b} = (b_i, 1 \le i \le m) \subset P_+ \ni c$, we set: $P_{\mathbf{b}} \stackrel{\mathbf{def}}{=\!=} \prod_i P_{b_i}$, $\mathbb{C}^{cu}_{\mathbf{b}} =$

$$\frac{\langle P_{\mathbf{b}} P^{\iota}_c \theta_u \mu\rangle}{\langle P_c P^{\iota}_c \mu\rangle} = \frac{\langle P_{\mathbf{b}} P_{c^{\iota}} \theta \mu\rangle}{u(\Sigma_i b_i - c)\, \langle P_c P^{\iota}_c \mu\rangle} = \frac{\dot{\tau}_-^{-1}(P_{\mathbf{b}} P_{c^{\iota}})(q^{\rho_k})\langle \theta\mu\rangle}{u(\sum_i b_i - c)\, \langle P_c P^{\iota}_c \mu\rangle},$$

where $\dot{\tau}_-(P_b) = q^{-(b,b)/2-(b,\rho_k)} P_b$ for $b \in P_+$ is the action of τ_- in the polynomial representation. The extension to any ℓ is as above.

THM. For A_n and $u = 1$, there exists a unique series $\mathfrak{C}(q,t,a)$ such that $\mathbb{C}^{0}_{00}/\langle\theta\mu\rangle = \mathfrak{C}(q,t,a = -t^{n+1})$ (stabilization). Then $\mathfrak{C}(q,t,a) = \mathfrak{C}(t^{-1}, q^{-1}, a)$ (superduality)

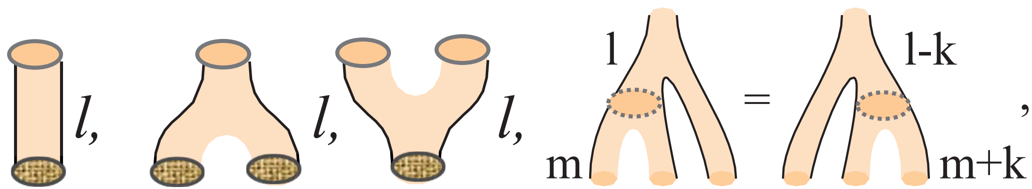
Here $\dot{\tau}_-^{-1}(P_{\mathbf{b}} P_{c^{\iota}})(q^{\rho_k})/P_c(q^{\rho_k})$ is the DAHA-Jones "polynomial" from [*Ch,Danilenko,2015*] for Hopf $(m{+}1)$-link with the pairwise linking numbers -1 for colors $\mathbf{b}$ and $+1$ between $\mathbf{b}$ and c.

ASSOCIATIVITY VIA TQFT

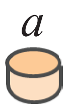
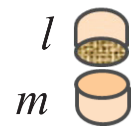
Following $TQFT$ (the unoriented one due to Turaev-Tuner with ι), the relations between $\mathbb{C}_{\mathbf{b}}^{c\mathbf{u}}$ can be interpreted as follows. Let $\mathscr{A}$ be a commutative algebra with 1 and a symmetric non-degenerate form $\langle f,g\rangle=\langle fg^{\iota}\mu_1\rangle$ for $\epsilon:\mathscr{A}\ni f\mapsto\langle f\mu_1\rangle$, $\mu_1^{\iota}=\mu_1$, $1^{\iota}=1,\epsilon(1)=1$. Define $\Delta:\mathscr{A}\to\mathscr{A}\widehat{\otimes}\mathscr{A}$ via $\langle\Delta(f),x\otimes y\rangle=\langle f,xy\rangle$. In the basis of orthogonal polynomials/functions $\{P_a\in\mathscr{A}\}$ under $P_0=1,\langle 1,1\rangle=1$: $\Delta(P_aV)=\sum_{b,c}\frac{\langle P_aV,P_bP_c\rangle P_b\otimes P_c}{\langle P_b,P_b\rangle\langle P_c,P_c\rangle}$ for any ι-invariant function V. The invariant of S^2 is then $\langle V\mu_1\rangle$. Taking $V=\theta_{\mathbf{u}}^{(\ell)}$, $P_a(a\in P_+)$ etc., as above, it is $\langle\theta_{\mathbf{u}}^{(\ell)}\mu\rangle/\langle\mu\rangle$. The corresponding invariant for the torus T^2 is $\sum_{b\in P_+}\frac{\langle\theta_{\mathbf{u}}^{(\ell)},P_bP_b\rangle}{\langle P_b,P_b\rangle}$. For $A_1,\theta_{\mathbf{u}}^{(\ell)}=\theta$ as $t\to 0$, it is proportional to $1+\sum_{m\geq 1}\frac{1}{(1-q)\cdots(1-q^m)}$, which diverges as $|q|<1$. One can use here some renormalization (and analytic continuation), roots of unity q, ... or proper V. There are no convergence problems though for $\theta_{\mathbf{u}}^{(\ell)}$ $(\ell\geq 0)$ if no "cycles" are allowed (the next page)!

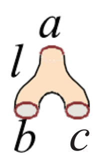
TQFT WITH LEVELS

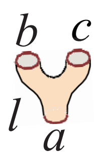
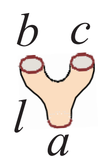
Generators, relations and some amplitudes:



,

$\vartheta^l = \sum l$  P_a, $\vartheta^l P_a = l$  $1 + \ldots$,  $= \dfrac{\langle \vartheta^{l+m} \mu \rangle}{\langle \mu \rangle}$,

where $P_0 = 1$,  $= \dfrac{\langle P_a P_b^{\iota} P_c^{\iota} \vartheta^l \mu \rangle}{\langle P_a P_a^{\iota} \mu \rangle} = \dfrac{\langle P_a^{\iota} P_b P_c \vartheta^l \mu \rangle}{\langle P_a P_a^{\iota} \mu \rangle}$. Δ :

$P_a \vartheta^l \to \sum_{\{b,c\}}$  $P_b \otimes P_c$,  $= \dfrac{\langle P_a P_b^{\iota} P_c^{\iota} \vartheta^l \mu \rangle \langle \mu \rangle}{\langle P_b P_b^{\iota} \mu \rangle \langle P_c P_c^{\iota} \mu \rangle}$.

NIL-THEORY: THE LIMIT $t \to 0$

The usual Rogers-Ramanujan sums occur as $t \to 0$ ($t_\nu \to 0$, to be exact). The μ-function and P-polynomials are well-defined at $t=0$; we put then $\bar\mu, \bar P_b, \bar{\mathbb{C}}^{\cdots}_{\cdots}$. Using $\lim_{t\to 0} q^{(b,\rho_k)} P_b^\iota(q^{c+\rho_k}) = q^{-(b,c)}$, one obtains: $\bar{\mathbb{C}}^{cu}_{0b} \stackrel{\text{def}}{=\!=} \frac{\langle \bar P_b \bar P_c^\iota \theta_u\, \bar\mu\rangle}{\langle \bar P_c \bar P_c^\iota \bar\mu\rangle} = \frac{q^{(b-c)^2/2}}{u(b-c)\prod_{i=1}^n \prod_{j=1}^{(c,\alpha_i^\vee)}(1-q_i^j)}$,

$$\bar{\mathbb{C}}^{c\mathbf{u}}_{0b} = \sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \bar{\mathbb{C}}^{c_1 u_1}_{0b}\, \bar{\mathbb{C}}^{c_2 u_2}_{0c_1}\, \bar{\mathbb{C}}^{c_3 u_3}_{0c_2} \cdots \bar{\mathbb{C}}^{c u_\ell}_{0,c_{\ell-1}} \quad \textbf{(R-R)}$$

$= \sum_{c_1,c_2,\ldots,c_{\ell-1}} \frac{q^{(c_0-c_1)^2/2+(c_1-c_2)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell} u_p(c_{p-1}-c_p)\prod_{i=1}^n \prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)}$, **where** $c_i \in P_+, q_i = q_{\alpha_i}, \alpha_i^\vee = 2\alpha_i/(\alpha_i,\alpha_i)$**, and we set** $c_0 = b, c_\ell = c \in P_+$**.**

Here q-Hermite polynomials $\bar P_b$ coincide with dominant Demazure level-one characters (*Sanders, Ion*). Upon the division by their norms, they coincide with the characters of some natural quotients of the upper level-one Demazure modules and those of global Weyl modules.

RELATION TO STRING FUNCTIONS

Let us discuss briefly the connections with string functions. Here $\widehat{\theta}_v(X) \stackrel{\mathbf{def}}{=\!=} \sum_{a\in v+Q} q^{\frac{(a,a)}{2}} X_a$ **for** $v \in P/Q$ **are more convenient. Then the corresponding** $\langle \bar{P}_b \bar{P}_c^{\iota} \widehat{\theta}_{\mathbf{v}} \bar{\mu}\rangle / \langle \bar{P}_c \bar{P}_c^{\iota} \bar{\mu}\rangle$ **for** $c_0 = b, c_\ell = c$ **are**

$$\widehat{\mathbb{C}}^{\mathbf{v}}_{b,c} = \sum_{c_1,c_2,\ldots,c_{\ell-1}\in P_+} \frac{q^{(c_0-c_1)^2/2+\ldots+(c_{\ell-1}-c_\ell)^2/2}}{\prod_{p=1}^{\ell}\prod_{i=1}^{n}\prod_{j=1}^{(c_p,\alpha_i^\vee)}(1-q_i^j)},$$ **where** $\mathbf{v} = \{v_1,\ldots,v_\ell\} \subset P/Q$ **and the summation is over** $c_i - c_{i+1} \in v_i + Q$**. They are zero unless** $b - c + v_1 + \ldots + v_\ell \in Q$**. When** $b = 0$**, they are modular weight-zero functions for minuscule** c**, w.r.t. some congruence subgroups of** $SL(2,\mathbb{Z})$ **and up to** $q^\bullet$**. Let** $\eta = q^{\frac{1}{24}} \prod_{i=1}^{\infty}(1-q^i)$**.**

First, $q^{-\frac{1}{4}} \widehat{\mathbb{C}}^{111}_{0,1} = \prod_{j=1}^{\infty}(1+q^j)^2 \sum_{m=0}^{\infty} \frac{q^{2m^2}}{\prod_{j=1}^{m}(1-q^{2j})}$ **for** A_1 **and** $\ell = 3$; $\sum_{m=0}^{\infty}$ **is the Rogers-Ramanujan "G" after** $q^2 \mapsto q$**. Upon** $\frac{q^\bullet}{\eta^2}\times$, $\widehat{\mathbb{C}}^{000}_{0,0}, \widehat{\mathbb{C}}^{110}_{0,0}, \widehat{\mathbb{C}}^{100}_{0,1}, \widehat{\mathbb{C}}^{111}_{0,1}$ **coincide with the basic string functions for** $\widehat{sl_3}$ **of level 2:** $C_0^{2\widehat{\omega}_0}, C_{\alpha_1+\alpha_2}^{2\widehat{\omega}_0}, C_{\omega_1}^{\widehat{\omega}_0+\widehat{\omega}_1}, C_{\omega_1+\alpha_2}^{\widehat{\omega}_0+\widehat{\omega}_1}$ [*Georgiev, 1995*].

LEVEL-RANK DUALITY

Here $\widehat{\lambda} = \lambda + \delta$ for $\lambda \in P_+$, $\omega_0 = 0$, and string functions for affine dominant Λ of level ℓ are the coefficients of the decomposition of the character of the integrable Kac-Moody module L_Λ in terms of the standard affine orbit sums ϑ^ℓ_ν; namely, $\chi(L_\Lambda) = \sum_\nu C^\Lambda_\nu \vartheta^\ell_\nu$.

The calculations are quite involved here (based on parafermions). Thus we arrived at the level-rank duality (*I.Frenkel and others*) for certain string functions. Surprisingly, this duality is simple to observe in terms of the sums $\widehat{\mathbb{C}}$. The quadratic q-powers here are given in terms of the (inverse) Cartan matrix for the root system $R \otimes A_{\ell-1}$. So for $R = A_{n-1}$, a straightforward analysis shows that they satisfy $n \leftrightarrow \ell$. At the level of sets v: the ℓ-sets of the element from $P/Q = \mathbb{Z}_n$ for A_{n-1} are naturally identified with n-sets of the elements from $P/Q = \mathbb{Z}_\ell$ for $A_{\ell-1}$. Note that counting classes of integrable modules, you have essentially $\binom{n+\ell-1}{n-1}/n = \binom{n+\ell-1}{\ell-1}/\ell$, but the duality for the corresponding string functions is generally much more subtle.

NONSYMMETRIC THEORY

Following [*Ch, Kato, 2018*]**, we outline the identification of the non-symmetric global Weyl modules** [*E.Feigin, Kato, Khoroshkin, Macedonskiy,...*)] **with the Demazure slices of the upper Demazure filtration in the (basic) level-one module L. The upper Demazure modules are with respect to $\widehat{\mathfrak{b}}_-$ in contrast to the Borel subalgebra $\widehat{\mathfrak{b}}_+$, resulting in the usual level-one Demazure modules D_b, $b \in P$. The characters of the latter coincide with non-symmetric q-Hermite polynomials $\bar{E}_b = E_b(t \to 0)$** (*Sanderson, Ion*)**, where E_b are nonsymmetric Macdonald polynomials for $b \in P$. They are orthogonal for the same μ, but now form a basis in the whole $\mathbb{C}[X_b]$. The characters of Demazure slices are identified with $E_b^\dagger = E_b(t \to \infty)$, divided by their norms h_b^0, which can be defined as the limits $t \to 0$ of the norms of E_b. The dag-polynomials are significantly more subtle than $\bar{E}_b$, though $P_b^\dagger$ are closely related to $\bar{P}_b$ (for $b \in P_+$). Let us relate the decomposition of $L^{\otimes \ell}$ via the Demazure slices to R-R sums.**

DEMAZURE SLICES

The first part is entirely numerical (based on the DAHA theory). Let $\widehat{\theta} \stackrel{\mathbf{def}}{=\!=} \theta \frac{\langle\bar{\mu}\rangle}{\langle\theta\bar{\mu}\rangle}$, $\bar{\mu} = \mu(t \to 0)$ (actually, $\langle\theta\bar{\mu}\rangle = 1$); then $\widehat{\theta}$ can be identified with the graded character of the level-one (basic) integrable representation L of the twisted affinization $\widehat{\mathfrak{g}}$ of the simple Lie algebra $\mathfrak{g}$ corresponding to the root system R.

For $\ell \in \mathbb{N}, b \in P$ and $\mathbf{c} = \{c_i \in P,\ 1 \le i \le \ell\}$, $\quad \bar{E}_{b^\iota}\,\widehat{\theta}^\ell = \sum_{\mathbf{c}} C_{\mathbf{c}}\, \frac{q^{\bigl((b_+-(c_1)_+)^2+\ldots+((c_{\ell-1})_+-(c_\ell)_+)^2\bigr)/\,2}}{\prod_{i=1}^{\ell-1}\ h^0_{c_i}}\ \frac{E^{\dagger *}_{c_\ell}}{h^0_{c_\ell}}$, where $C_{\mathbf{c}}$ is some (non-trivial) power of q, $E^{\dagger *}_c$ is $E^{\dagger}_c$ where $X_a \to X_a^{-1}, q \to q^{-1}$.

Its Kac-Moody interpretation is essentially as follows. For a level one usual Demazure module D_b associated to $b \in P$ and its dual $D_b^\vee$, the module $D_b^\vee \otimes L^{\otimes \ell}$ admits a filtration by the Demazure slices (as constituents). Its multiplicities are provided by the formula above. This can be (and was) generalized in various directions.