The deformation formula of quantum dynamical Yang-Baxter equations over a non-abelian base

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Representation Theory, Integrable Systems and Related Topics

BIMSA

11 July 2024

joint work with Zhuo Chen in Tsinghua University

Background

A formal star product on a Poisson manifold M is a formal associative deformation of the algebra of smooth functions $C^\infty(M)$.

The existence and classification of star products has been widely studied.

- Symplectic case: [De Wilde-Lecomte]; [Fedosov]; ...,
- Poisson case: [Kontsevich]; [Cattaneo-Felder]; ... .

Drinfel'd showed that the quantization problem for the triangular Lie bialgebras is equivalent to constructing a universal deformation formula [Drinfel'd].

This provides an approach to construct special star products: Suppose that M is a manifold with an action of a Lie algebra $\mathfrak{g}$, and

$$F \in U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$$

is a Drinfel'd twist, i.e., F satisfies

$$\begin{aligned} (\Delta \otimes \mathrm{id})F \cdot F_{12} &= (\mathrm{id} \otimes \Delta)F \cdot F_{23}\,; \\ (\epsilon \otimes \mathrm{id})F &= (\mathrm{id} \otimes \epsilon)F = 1\,; \\ F &= 1 \otimes 1 + O(\hbar)\,. \end{aligned}$$

Write $F = F_1 \otimes F_2\,, F_1\,, F_2 \in U(\mathfrak{g})$ (Sweedler's notation). Then

$$f \star g := F(f\,, g) = (F_1 \triangleright f) \cdot (F_2 \triangleright g)\,, \quad \forall f\,, g \in C^\infty(M)[\![\hbar]\!]\,,$$

defines a star product on $C^\infty(M)[\![\hbar]\!]$.

Background

An iterative procedure to write down an explicit universal deformation formula starting from classical triangular r-matrices is given in the work [Dolgushev-Isaev-Lyakhovich-Sharapov].

This approach extends the Fedosov approach and has the advantage of not choosing auxiliary symplectic connections, it provides star products for a broad class of manifolds with irregular Poisson brackets.

In this talk, we will extend the method of [Dolgushev-Isaev-Lyakhovich-Sharapov] to the case of classical triangular dynamical r-matrices.

Background

Xu introduced the quantum dynamical Yang–Baxter equation (QDYBE), or generalized Gervais–Neveu–Felder equation, over a nonabelian base $\mathfrak{h}^*$.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ be a not necessarily abelian Lie subalgebra.

Let $\{h_i\}$ be a vector basis of $\mathfrak{h}$, $\{\lambda^i\}$ be the corresponded linear coordinates on $\mathfrak{h}^*$, and $\{\xi^i \in \mathfrak{h}^*\}$ be the dual basis.

Let

$$R : \mathfrak{h}^* \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$$

be a $\mathfrak{h}$-equivariant map, i.e., the following zero weight condition is satisfied

$$(L_{\mathrm{Ad}^*_{h_i}} R)(\lambda) + [h_i \otimes 1 + 1 \otimes h_i \, , R(\lambda)] = 0 \, , \quad \forall h_i \in \mathfrak{h} \, , \lambda \in \mathfrak{h} \, .$$

We treat R as an element in $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \otimes C^\infty(\mathfrak{h}^*)[\![\hbar]\!]$.
The space $C^\infty(\mathfrak{h}^*)[\![\hbar]\!]$ is endowed with the PBW product $\star_{\mathrm{PBW}}$, induced by the PBW isomorphism $S(\mathfrak{h}) \simeq U(\mathfrak{h})$.

The non-abelian QDYBE has the following form:

$$\begin{aligned} R_{12}(\lambda) \star_{\mathrm{PBW}} R_{13}(\lambda + \hbar h^{(2)}) \star_{\mathrm{PBW}} R_{23}(\lambda) \\ = R_{23}(\lambda + \hbar h^{(1)}) \star_{\mathrm{PBW}} R_{13}(\lambda) \star_{\mathrm{PBW}} R_{12}(\lambda + \hbar h^{(3)}) \end{aligned}$$

Background

The classical limit of the quantum dynamical R-matrix is the classical dynamical r-matrix,

$$r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g} \,,$$
$$r(\lambda) := \lim_{\hbar \to 0} \frac{1}{\hbar}(R(\lambda) - 1 \otimes 1) \,,$$

satisfies the $\mathfrak{h}$-equivariant condition

$$(L_{\mathrm{Ad}^*_{h_i}} r)(\lambda) + [h_i \otimes 1 + 1 \otimes h_i \,, r(\lambda)] = 0$$

and the non-abelian classical dynamical Yang-Baxter equation(CDYBE)

$$\sum_i h_i \frac{\partial}{\partial \lambda_i} r + [r \,, r] = 0 \,.$$

Background

Suppose that

$$F : \mathfrak{h}^* \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$$

is a $\mathfrak{h}$-equivariant map and satisfies the nonabelian dynamical twist equation:

$$\begin{aligned}(\Delta \otimes \mathrm{id})F(\lambda) \star_{\mathrm{PBW}} F_{12}(\lambda + \hbar h^{(3)}) &= (1 \otimes \Delta)F(\lambda) \star_{\mathrm{PBW}} F_{23}(\lambda) \\ (\epsilon \otimes \mathrm{id})F(\lambda) &= (\mathrm{id} \otimes \epsilon)F(\lambda) = 1\,; \\ F(\lambda) &= 1 \otimes 1 + O(\hbar)\,.\end{aligned}$$

Then

$$R(\lambda) := F_{21}^{-1}(\lambda) \star_{\mathrm{PBW}} F(\lambda)$$

is a solution of the non-abelian QDYBE.

Background

Let G be a Lie group of $\mathfrak{g}$.

Theorem (Xu)

A map $r : \mathfrak{h}^* \to \mathfrak{g} \wedge \mathfrak{g}$ satisfies the (non-abelian) CDYBE if and only

$$\pi_r := \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \overrightarrow{h_i} + \overrightarrow{r(\lambda)}$$

is a Poisson structure on $\mathfrak{h}^* \times G$.
Here $\pi_{\mathfrak{h}^*} = \sum c_{i,j}^k \lambda^k \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \lambda_j}$ is the Lie-Poisson structure on $\mathfrak{h}^*$, $c_{i,j}^k$ are the structure constants of $\mathfrak{h}$.

We call $r(\lambda)$ a (non-abelian) symplectic triangular dynamical r-matrix, if π_r comes from a symplectic structure on $\mathfrak{h}^* \times G$.

Background

Theorem (Xu)

Any abelian (i.e., $\mathfrak{h}$ is abelian) symplectic or splittable triangular dynamical r-matrix is quantizable.

Theorem (Alekseev-Calaque)

Any non-abelian symplectic triangular dynamical r-matrix is quantizable.

Both Xu and Alekseev-Calaque adopted Fedosov approach to construct quantizations.

Background

Theorem (Xu)

A map $F : \mathfrak{h}^\star \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$ satisfies the (non-abelian) dynamical twist equation if and only if there is a star product $\star$ on $\mathfrak{h}^* \times G$ such that it satisfies the following conditions:

(1) $g_1(\lambda) \star g_2(\lambda) = g_1(\lambda) \star_{\mathrm{PBW}} g_2(\lambda)$, $\forall g_1(\lambda), g_2(\lambda) \in C^\infty(\mathfrak{h}^*)$;

(2) $f(x) \star g(\lambda) = f(x) \cdot g(\lambda)$, $\forall f(x) \in C^\infty(G), g(\lambda) \in C^\infty(\hbar^*)$;

(3) $g(\lambda) \star f(x) = \sum_k \frac{\hbar^k}{k!}(\frac{\partial^k}{\partial \lambda^{i_1} \dots \lambda^{i_k}} g) \cdot (\overrightarrow{h_{i_1}} \cdots \overrightarrow{h_{i_k}} g)$, $\forall f(x) \in C^\infty(G), g(\lambda) \in C^\infty(\mathfrak{h}^*)$;

(4) $f_1(x) \star f_2(x) = \overrightarrow{F(\lambda)}(f_1, f_2)$, $\forall f_1(x), f_2(x) \in C^\infty(G)$.

Such star product is called a compatible star product. If it exists, it is completely determined by F in terms of an explicit formula (Xu). We note that a compatible star product is not symmetric with respect to variables $\lambda \in \mathfrak{h}^*$ and $x \in G$. Thus it can not be of Weyl ordering.

Questions

When $\mathfrak{h}$ is abelian, consider the Lie algebroid

$$(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*} := \left(T\mathfrak{h}^* \oplus \mathfrak{g} \to \mathfrak{h}^*\right)$$

over the base manifold $\mathfrak{h}^*$. Here $\mathfrak{g}$ acts trivially on $\mathfrak{h}^*$. In this abelian case, Xu construct a twistor

$$\mathcal{F} := F(\lambda) \cdot \Theta \in U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*} \otimes_{C^\infty(\mathfrak{h}^*)} U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}[\![\hbar]\!]\,,$$

where

$$\Theta = \exp(\hbar\theta)\,, \quad \theta = \sum_i \frac{\partial}{\partial \lambda_i} \otimes h_i\,, \quad h_i \in \mathfrak{h} \subset \mathfrak{g}\,.$$

From this twistor $\mathcal{F}$, Xu constructed the quantum groupoid

$$\left(U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}[\![\hbar]\!]\,, \cdot\,, \Delta_{\mathcal{F}}\,, \alpha_{\mathcal{F}}\,, \beta_{\mathcal{F}}\,, \epsilon\right), \quad \Delta_{\mathcal{F}}^{\mathrm{op}}(x) = R(\lambda) \cdot \Delta(x) \cdot R^{-1}(\lambda)$$

which corresponds to the quantum dynamical R-matrix $R(\lambda) = F_{21}(\lambda)^{-1} F(\lambda)$.

Question 1:
When $\mathfrak{h}$ is a non-abelian Lie subalgebra of $\mathfrak{g}$, and $R(\lambda)$ is a non-abelian quantum dynamical R-matrix, what is the quantum groupoid (in the sense of Xu) corresponds to $R(\lambda)$, what is the twistor $\mathcal{F}$ for constructing this quantum groupoid?

Question 2:
Suppose that the Lie algebra $\mathfrak{g}$ acts on a manifold M. What is the universal deformation formula for the aforesaid quantum groupoid? Suppose that $R(\lambda)$ is a quantization of a symplectic triangular dynamical r-matrix $r(\lambda)$. Can we have an iterative procedure to write down an explicit universal deformation formula starting from $r(\lambda)$, which generalizes the results of [Dolgushev-Isaev-Lyakhovich-Sharapov]?

The field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let R be an associative algebra with unit.

Definition (following Xu's Quantum Groupoid paper)

A quantum groupid $(\mathcal{H}, \cdot, \Delta, \alpha, \beta, \epsilon)$ over R consists of the following data with compatibility conditions:

(1) $\mathcal{H}$ is an associative algebra with unit 1.

(2) Two associative algebra morphisms, the **source map** $\alpha : R \to \mathcal{H}$ and the **target map** $\beta : R^{\mathrm{op}} \to \mathcal{H}$. Their images commute.
The (R, R)-bimodule structure on $\mathcal{H}$:

$$r \cdot h := \alpha(r) \cdot h, \quad h \cdot r' := \beta(r') \cdot h, \quad \forall r, r' \in R, h \in \mathcal{H}.$$

(3) The **coproduct** $\Delta : \mathcal{H} \to \mathcal{H} \otimes_R \mathcal{H}$ is a coassociative (R, R)-bimodule morphism, satisfies $\Delta(1) = 1 \otimes 1$, and

$$\Delta(h) \cdot (\beta(r) \otimes 1 - \alpha(r) \otimes 1) = 0, \quad \forall h \in \mathcal{H}, r \in R,$$

$$\Delta(h_1 \cdot h_2) = \Delta(h_1) \cdot \Delta(h_2), \quad \forall h_1, h_2 \in \mathcal{H}.$$

Definition (continuted)

(4) The **counit map** $\epsilon : \mathcal{H} \to R$, a (R, R)-bimodule morphism, satisfies $\epsilon(1) = 1$, $(\epsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id}$, and

$$\mathrm{Ker}(\epsilon) \text{ is a left } \mathcal{H}\text{-ideal in } \mathcal{H}.$$

Example

Suppose that $L \to M$ is a Lie algebroid over M. Then the universal enveloping algebra $U(L)$ is a quantum groupoid over $C^\infty(M)$, with $\alpha = \beta : C^\infty(M) \to U(L)$ the inclusion map.

Let $(\mathcal{H}\,,\cdot\,,\Delta\,,\alpha\,,\beta\,,\epsilon)$ be a quantum groupoid over R.

Definition

An element $\mathcal{F} \in \mathcal{H} \otimes_R \mathcal{H}$ is called a twistor, if it satisfies the following conditions:

(1) $(\Delta \otimes \mathrm{id})\mathcal{F} \cdot \mathcal{F}_{12} = (\mathrm{id} \otimes \Delta)\mathcal{F} \cdot \mathcal{F}_{23}\,, \quad (\epsilon \otimes \mathrm{id})\mathcal{F} = (\mathrm{id} \otimes \epsilon)\mathcal{F} = 1\,.$

(2) For all left $\mathcal{H}$-modules M_1 and M_2, the maps

$$\begin{aligned} \mathcal{F}^\sharp : M_1 \otimes_{R_\mathcal{F}} M_2 &\to M_1 \otimes_R M_2 \\ m_1 \otimes_{R_\mathcal{F}} m_2 &\mapsto \mathcal{F} \cdot (m_1 \otimes_R m_2) \end{aligned}$$

are all isomorphisms.

In Condition (2) above, $R_\mathcal{F}$ denotes a new associative algebra structure on R whose multiplication is given by

$$r_1 \star_\mathcal{F} r_2 := (\mathcal{F}_1 \triangleright r_1) \cdot (\mathcal{F}_2 \triangleright r_2)\,, \quad \forall r_1\,, r_2 \in R\,.$$

Condition (1) guarantees that $\star_\mathcal{F}$ is an associative multiplication.

Twistors

Theorem (Xu)

If $\mathcal{F} \in \mathcal{H} \otimes_R \mathcal{H}$ is a twistor, then we have a new quantum groupoid

$$\mathcal{H}_{\mathcal{F}} = \left(\mathcal{H}\,, \cdot\,, \Delta_{\mathcal{F}}\,, \alpha_{\mathcal{F}}\,, \beta_{\mathcal{F}}\,, \epsilon\right),$$

over the deformed associative algebra $R_{\mathcal{F}} = (R\,, \star_{\mathcal{F}})$. The multiplication $\cdot$ and counit ϵ of $\mathcal{H}_{\mathcal{F}}$ are unchanged, and

$$\begin{aligned} \Delta_{\mathcal{F}}(x) &= (\mathcal{F}^{\sharp})^{-1}(\Delta(X) \cdot \mathcal{F})\,, \quad \forall x \in \mathcal{H}; \\ \alpha_{\mathcal{F}}(r) &= (\mathcal{F}_1 \triangleright r)\mathcal{F}_2\,, \quad \beta_{\mathcal{F}}(r) = \mathcal{F}_1(\mathcal{F}_2 \triangleright r)\,, \\ r_1 \star_{\mathcal{F}} r_2 &= (\mathcal{F}_1 \triangleright r_1)(\mathcal{F}_2 \triangleright r_2)\,, \qquad \forall r\,, r_1\,, r_2 \in R\,. \end{aligned}$$

Suppose that $\mathfrak{h} \subset \mathfrak{g}$ is a non-abelian Lie subalgebra.
In this case, $\mathfrak{h}$ acts non-trivially on $\mathfrak{h}^*$ by the adjoint action.
Meanwhile, the abelian Lie algebra $\mathfrak{h}^*$ acts on the manifold $\mathfrak{h}^*$ by derivations with constant coefficients:

$$\xi^i \in \mathfrak{h}^* \;\mapsto\; \frac{\partial}{\partial \lambda^i} \in \Gamma(T\mathfrak{h}^*)\,.$$

The pair $(\mathfrak{h}\,,\mathfrak{h}^*)$ is a Lie bialgebra, its Drinfel'd double is denoted by $\mathbb{D}(\mathfrak{h})$. The aforesaid two actions of $\mathfrak{h}$ and $\mathfrak{h}^*$ on the manifold $\mathfrak{h}^*$ extends to a Lie algebra action of $\mathbb{D}(\mathfrak{h})$ on $\mathfrak{h}^*$.

Let $\mathbb{D}(\mathfrak{h})_{\mathfrak{h}^*} := \Big(\mathbb{D}(\mathfrak{h}) \times \mathfrak{h}^* \to \mathfrak{h}^*\Big)$ be the action Lie algebroid of $\mathbb{D}(\mathfrak{h})$ over $\mathfrak{h}^*$.

We have the identification of the two Lie algebroids over $\mathfrak{h}^*$

$$\mathbb{D}(\mathfrak{h})_{\mathfrak{h}^*} = (T\mathfrak{h}^* \oplus \mathfrak{h})_{\mathfrak{h}^*}\,.$$

Let H be a Lie group of $\mathfrak{h}$.
The canonical classical r-matrix

$$\sum_i \xi^i \wedge h_i \quad \in \wedge^2 \mathbb{D}(\mathfrak{h})$$

induces the Poisson structure

$$\sum_i \frac{\partial}{\partial \lambda^i} \wedge L_{\mathrm{Ad}^*_{h_i}} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i} = \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda^i} \wedge \overrightarrow{h_i}$$

on $\mathfrak{h}^* \times H$. It is known that this Poisson structure corresponds to the canonical symplectic structure on the cotangent bundle

$$T^*H \simeq \mathfrak{h}^* \times H\,.$$

In [Gutt], a compatible star product which quantizes the above symplectic structure is given, thus we have the corresponded twistor

$$\widetilde{\Theta} \in U(T\mathfrak{h}^* \oplus \mathfrak{h})_{\mathfrak{h}^*} \otimes_{C^\infty(\mathfrak{h}^*)} U(T\mathfrak{h}^* \oplus \mathfrak{h})_{\mathfrak{h}^*}\,.$$

In the following, we will assume that $\mathfrak{g}$ admits a reductive decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}\,, \qquad [\mathfrak{h}\,, \mathfrak{m}] \subset \mathfrak{m}\,.$$

We can extend the adjoint $\mathfrak{h}$-action on $\mathfrak{h}^*$ to an action of $\mathfrak{g}$ on $\mathfrak{h}^*$, by setting the action of $\mathfrak{m}$ on $\mathfrak{h}^*$ to be zero.
Then we obtain a Lie algebroid

$$(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*} := \left(T\mathfrak{h}^* \oplus \mathfrak{g} \to \mathfrak{h}^*\right)$$

which contains the Lie subalgebroid $(T\mathfrak{h}^* \oplus \mathfrak{h})_{\mathfrak{h}^*}$.

Theorem

Suppose that $\mathfrak{g}$ admits a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (So that $(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}$ is defined). Let $F : \mathfrak{h}^* \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$ be a map which has the form $F(\lambda) = 1 \otimes 1 + O(\hbar)$. Then

$$\widetilde{\mathcal{F}} := F(\lambda) \cdot \widetilde{\Theta} \in U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*} \otimes_{C^\infty(\mathfrak{h}^*)} U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}[\![\hbar]\!]$$

is a twistor of the quantum groupoid $U(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}[\![\hbar]\!]$ if and only if $F(\lambda)$ is a solution of the non-abelian dynamical twist equation.

The proof relies on the results of [Donin-Murdov] on twisting by dynamical cocycles.

Corollary

With the same assumption as above. Let M be a manifold with an action of a Lie algebra $\rho : \mathfrak{g} \to TM$. Then the universal deformation formula

$$f(\lambda, x) \star g(\lambda, x) := ((\rho \otimes \rho)\widetilde{\mathcal{F}})(f, g), f, g \in C^{\infty}(\mathfrak{h}^* \times M),$$

for all $f, g \in C^{\infty}(\mathfrak{h}^* \times M)$, $\lambda \in \mathfrak{h}^*$, $x \in M$,
defines a star product on $\mathfrak{h}^* \times M$ which satisfies the following properties:

(1) $g_1(\lambda) \star g_2(\lambda) = g_1(\lambda) \star_{\mathrm{PBW}} g_2(\lambda)$, $\forall g_1(\lambda), g_2(\lambda) \in C^{\infty}(\mathfrak{h}^*)$;

(2) $f(x) \star g(\lambda) = f(x) \cdot g(\lambda)$, $\forall f(x) \in C^{\infty}(M), g(\lambda) \in C^{\infty}(\hbar^*)$;

(3) $g(\lambda) \star f(x) = \sum_k \frac{\hbar^k}{k!}(\frac{\partial^k}{\partial \lambda^{i_1} \dots \lambda^{i_k}} g) \cdot (\rho(h_{i_1}) \cdots \rho(h_{i_k}) g)$, $\forall f(x) \in C^{\infty}(M), g(\lambda) \in C^{\infty}(\mathfrak{h}^*)$;

(4) $f_1(x) \star f_2(x) = (\rho \otimes \rho)(\overrightarrow{F(\lambda)})(f_1, f_2)$, $\forall f_1(x), f_2(x) \in C^{\infty}(M)$.

Deformation Formula from Non-abelian r-matrices

In the following we will assume that $r : \mathfrak{h}^* \to \mathfrak{g} \wedge \mathfrak{g}$ is a symplectic triangular dynamical r-matrix. We know that $r(\lambda)$ can be quantized to a quantum dynamical twist $F(\lambda) : \mathfrak{h}^* \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$.

Since $\pi_r := \pi_{\mathfrak{h}^*} + \sum_i \frac{\partial}{\partial \lambda_i} \wedge \overrightarrow{h_i} + \overrightarrow{r(\lambda)} \in \Gamma(\wedge^2(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*})$ is non-degenerate, we obtain the symplectic from

$$\omega_r := \pi_r^{-1} : \wedge^2(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*} \to \mathbb{K}\,.$$

This 2-cocycle defines a Lie algebroid extension

$$0 \to \mathbb{K}\hbar \to L \to (T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*,\hbar} \to 0$$

over $\mathfrak{h}^*$. We use $\{y_\alpha\}$ to denote the basis of fiber vectors of $(T\mathfrak{h}^* \oplus \mathfrak{g})_{\mathfrak{h}^*}$ over $\mathfrak{h}^*$. Then the Lie bracket on $\Gamma(L)$ has the form:

$$[y_\alpha\,, y_\beta]_L = \hbar[y_\alpha\,, y_\beta]_{T\mathfrak{h}^*\oplus\mathfrak{g}} + \hbar\omega_r(y_\alpha\,, y_\beta)\,,$$
$$[y_\alpha\,, f\hbar]_L = (y_\alpha \cdot f)\hbar\,, \quad \forall y_\alpha \in \Gamma(T\mathfrak{h}^* \oplus \mathfrak{g})\,,\ f \in C^\infty(\mathfrak{h}^*)\,.$$

Here $y_\alpha \cdot f$ means the action of y_α on the function $f \in C^\infty(\mathfrak{h}^*)$ by the anchor map.

Consider the universal enveloping algebra $U(L)$ of L over $C^\infty(\mathfrak{h}^*)$. It is endowed with the natural associative multiplication, denoted by $\circ$.

Let $\sigma : U(L) \otimes C^\infty(M) \to C^\infty(\mathfrak{h}^* \times M)[\![\hbar]\!]$ be the projection map which sends all homogeneous symmetric products of vectors y_α (as PBW basis) in $U(L)$ to zero.

Let M be a smooth manifold with a Lie algebra $\mathfrak{g}$-action, so that we have a morphism of Lie algebroid over $\mathfrak{h}^* \times M$:

$$\rho : L = (T\mathfrak{h}^* \oplus \mathfrak{g} \oplus \mathbb{K}\hbar)_{\mathfrak{h}^*} \to T(\mathfrak{h}^* \times M)\,.$$

Here the action of the trivial line bundle $\mathbb{K}\hbar$ on $\mathfrak{h}^* \times M$ is zero.

Define an operator $\delta : U(L) \otimes C^\infty(M) \to U(L) \otimes C^\infty(M)$:

$$\delta(A) := \sum_{\alpha\,,\beta} y_\beta (\omega_r)^{-1}_{\alpha\,,\beta} (\rho(y_\alpha) \cdot A + [y_\alpha\,,\, A]_\circ)\,, \quad \forall A \in U(L) \otimes C^\infty(M)\,.$$

Here $[y_\alpha\,,\,\cdot\,]_\circ$ is the commutator with respect the natural associative multiplication $\circ$ on $U(L)$.

Deformation Formula from Non-abelian r-matrices

Theorem

With the same assumption as above. Then the map

$$F : \mathfrak{h}^* \to U(\mathfrak{g}) \otimes U(\mathfrak{g})[\![\hbar]\!]$$

defined by the formula

$$\big((\rho \otimes \rho)F(\lambda)\big)(f, g) := \sigma\big(\exp(\hbar\delta)f \circ \exp(\hbar\delta)g\big)$$

for all $f, g \in C^\infty(M)$, satisfies the (non-abelian) quantum dynamical twist equation and its classical limit is the given symplectic triangular dynamical r-matrix $r(\lambda)$.

Here $\rho : \mathfrak{g} \to TM$ is the Lie algebra action, $exp(\hbar\delta)f, \exp(\hbar\delta)g \in U(L)$, and $exp(\hbar\delta)f \circ \exp(\hbar\delta)g$ is the multiplication in $U(L)$.

Thus we can further obtain the twistor $\widetilde{\mathcal{F}} := F(\lambda) \cdot \widetilde{\Theta}$, and the associated universal deformation formula.

Remark:

For general $f(\lambda\,,x)\,,g(\lambda\,,x)\in C^\infty(\mathfrak{h}^\star\times M)$, The formula

$$f\star_{\mathrm{Weyl}} g:=\sigma(\exp(\hbar\delta)f\circ\exp(\hbar\delta)g)$$

defines a star product, but it is not a compatible star product.

Even though, we can construct the desired quantum dynamical twist $F(\lambda)$ by setting

$$\big((\rho\otimes\rho)F(\lambda)\big)(f\,,g):=\sigma\big(\exp(\hbar\delta)f\circ\exp(\hbar\delta)g\big)\,,\forall f\,,g\in C^\infty(M)\,.$$

Then the desired compatible star product $\star$ is determined by $F(\lambda)=\sum f_{\alpha,\beta}(\lambda)U_\alpha\otimes U_\beta\,,f_{\alpha,\beta}(\lambda)\in C^\infty(\mathfrak{h}^*)\,,U_\alpha\,,U_\beta\in U(\mathfrak{g})[\![\hbar]\!]$, in terms of Xu's formula:

$$f(\lambda\,,x)\star g(\lambda\,,x)=$$

$$\sum\frac{\hbar^k}{k!}f_{\alpha,\beta}(\lambda)\star_{\mathrm{PBW}}\rho(U_\alpha)\frac{\partial^k f}{\partial\lambda^{i_1}\cdots\partial\lambda^{i_k}}\star_{\mathrm{PBW}}\rho(U_\beta)\rho(h_{i_1})\cdots\rho(h_{i_k})g\,,$$

for $\lambda\in\mathfrak{h}^*\,,x\in M$.

Thanks for your attention.