Affine Weyl groups and non-abelian discrete systems

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A short overview of the topic

Main examples: Painlevé equations.

- Problem: define new functions by an ODE of the m^{th} order with properties that generalize those of elliptic functions. [L. Fuchs], [H. Poincaré]
- Painlevé property: the general solution of an ODE has no critical movable points.
- $m = 2$: six classes defining the Painlevé transcendents. [Painlevé, 1902], [Gambier, 1910]

- Painlevé equations being one of the important objects in mathematics and mathematical physics have various non-commutative (NC) analogs: quantum [Nagoya, 2004], matrix differential [Kawakami, 2015], NC "differential" [Bobrova and Sokolov, 2023b], matrix difference [Cassatella-Contra et al., 2014], NC difference [Bobrova, Retakh, Rubtsov, Sharygin, 2024]
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- In the commutative case, discrete Painlevé equations have been studied in a series of papers by B. Gramaticos and A. Ramani since 1990s, but without understanding the whole picture.
- The latter was clarified by H. Sakai in his famous paper [Sakai, 2001], whose geometric method was inspired by a series of K. Okamoto's papers.
- NC birational geometry and Painlevé equations. [Okounkov and Rains, 2015], [Rains, 2019]
- An application of the affine Weyl groups to discrete Painlevé equations. [Noumi and Yamada, 1998], [Noumi and Yamada, 2000]
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Our goals & results

Global goal

- Generalize Sakai's approach to the non-commutative case in order to obtain a full classification of NC versions for the discrete Painlevé equations.

Local goal

- Generalize Noumi-Yamada's approach to the non-commutative case in order to obtain NC versions of the d-Painlevé equations.

Results

- Extended representations of the affine Weyl groups W and NC discrete systems.
- NC analogs for the d-Painlevé equations obtained by using extended representations of W. Note that the commutative degeneration scheme holds in the NC case.
- NC dressing chains in the Noumi-Yamada variables:
  - Lax pairs,
  - Bäcklund transformations,
  - a description of the related discrete dynamics.

A NC version of the d-P(E_7)

NC setting

- Consider an associative unital division ring $\mathcal{R}$ over the field $\mathbb{C}$ equipped with a derivation.
- We assume that all greek letters belong to the field $\mathbb{C}$, while the elements f_i are from $\mathcal{R}$. We will often call f_i as *functions*.
- The derivation $d_t : \mathcal{R} \to \mathcal{R}$ of the ring $\mathcal{R}$ is a $\mathbb{C}$-linear map satisfying the Leibniz rule. We also assume that there is a central element t such that $d_t(t) = 1$ and for any $\alpha \in \mathbb{C}$ we have $d_t(\alpha) = 0$. Here and below we identify the unit of the field with the unit of the ring.
- For the brevity we denote $d_t(f_i) = \dot{f}_i$, $d_t^2(f_i) = \ddot{f}_i$, and so on.
- Note that we have an involution on $\mathcal{R}$ called the transposition τ, which acts trivially on the generators of $\mathcal{R}$ and for any elements $F, G \in \mathcal{R}$ we have $\tau(F\,G) = \tau(G)\,\tau(F)$. This involution can be naturally extended to the matrices over $\mathcal{R}$.

Remark. We would rather not specify the generators of the ring $\mathcal{R}$ in order to avoid an overloaded description of a skew field (see [Cohn, 1995]). Instead, we encourage to think of the ring $\mathcal{R}$ as a generalization of rational functions over $\mathbb{C}$ to a non-commutative case.

The P_2 equation (1)

- Consider the P_2 system [Retakh and Rubtsov, 2010] (see also [Adler and Sokolov, 2021])

$$\begin{cases} \dot{q} &=& -q^2 + p - \frac{1}{2}t, \\ \dot{p} &=& qp + pq + \alpha_1. \end{cases} \qquad P_2$$

- Here we can assume that t is also an element of $\mathcal{R}$ such that $\dot{t} = 1$.
- Let $\alpha_0 + \alpha_1 = 1$ and $f := -p + 2q^2 + t$.
- Its Bäcklund transformations are given below (cf. with [Bershtein et al., 2023])

	α_0	α_1	q	p	t
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q - \alpha_0 f^{-1}$	$p - 2\alpha_0 q f^{-1} - 2\alpha_0 f^{-1} q + 2\alpha_0 f^{-2}$	t
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \alpha_1 p^{-1}$	p	t
π	α_1	α_0	$-q$	$-p + 2q^2 + t$	t

- These elements form an extended affine Weyl group of type $A_1^{(1)}$:

$$\widetilde{W}(A_1^{(1)}) = \langle s_0, s_1; \pi \rangle,$$
$$s_i^2 = 1, \qquad \pi^2 = 1, \qquad \pi s_i = s_{i+1}\pi, \qquad i \in \mathbb{Z}/2\mathbb{Z}. \tag{1}$$

The P_2 equation (2)

	α_0	α_1	q	p	t
s_0	$-\alpha_0$	$\alpha_1 + 2\alpha_0$	$q - \alpha_0 f^{-1}$	$p - 2\alpha_0 q f^{-1} - 2\alpha_0 f^{-1} q + 2\alpha_0 f^{-2}$	t
s_1	$\alpha_0 + 2\alpha_1$	$-\alpha_1$	$q + \alpha_1 p^{-1}$	p	t
π	α_1	α_0	$-q$	$-p + 2q^2 + t$	t

- Consider the translation operator $T = s_1\pi$. It acts on the parameters according to the formula below and forms a lattice on a line:

$$T(\alpha_0, \alpha_1) = (\alpha_0 - 1, \alpha_1 + 1). \tag{2}$$

- The q and p variables change as follows

$$\bar{q} = s_1\pi(q) = -s_1(q) = -q - \alpha_1 p^{-1}, \quad \bar{p} = s_1\pi(p) = s_1(-p + 2q^2 + t) = -p + 2\bar{q}^2 + t.$$

- So, we obtain the system for $T(q) = \bar{q}$, $T(p) = \bar{p}$

$$\begin{aligned} \bar{\alpha}_0 &= \alpha_0 - 1, & \bar{\alpha}_1 &= \alpha_1 + 1, \\ \bar{q} + q &= -\alpha_1 p^{-1}, & \bar{p} + p &= t + 2\bar{q}^2. \end{aligned} \qquad \text{d-P}(E_7)$$

- It generalizes to the non-commutative case the d-$\mathrm{P}(E_7)$ equation from [Sakai, 2001] (p. 206).
- It reduces to the following second-order difference equation for $T^n(q) = q_n$:

$$\alpha_{1,n}(q_{n+1} + q_n)^{-1} + \alpha_{1,n-1}(q_n + q_{n-1})^{-1} = -2q_n^2 - t, \quad \alpha_{1,n} = \alpha_1 + n. \qquad \text{alt-d-P}_1$$

The d-P(E_7) system: a Lax pair

$$\bar\alpha_0 = \alpha_0 - 1, \qquad \bar\alpha_1 = \alpha_1 + 1,$$
$$\bar q + q = -\alpha_1 p^{-1}, \qquad \bar p + p = t + 2\bar q^2. \qquad \text{d-P}(E_7)$$

- One may also consider the corresponding non-commutative discrete linear problem
$$\begin{cases} \partial_\lambda Y &= \mathcal{A}\, Y, \\ \bar Y &= \mathcal{B}\, Y. \end{cases} \tag{3}$$
- A Lax pair for the d-P(E_7) is given by
$$\mathcal{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & 1 \\ 2p & 0 \end{pmatrix} \lambda + \begin{pmatrix} -p + \frac{1}{2}t & -q \\ 2pq + 2\alpha_1 & p - \frac{1}{2}t \end{pmatrix},$$
$$\mathcal{B} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} -2q & -1 \\ -2\bar p & 0 \end{pmatrix},$$
where $t,\ \lambda \in \mathcal{Z}(\mathcal{R})$.
- Note that the compatibility condition is satisfied, since the commutator $[p, q]$ is invariant under the map
$$\psi : \mathcal{R}^2 \to \mathcal{R}^2, \quad (q, p) \mapsto (\bar q, \bar p) = \left(-p + t + 2q^2,\ -q - \bar\alpha_1(-p + t + 2q^2)^{-1}\right). \tag{4}$$
- Once $t \in \mathcal{R}$, the commutator $[p, q]$ is no longer a conserved quantity.

The d-P(E_7) system: a continuous limit

$$\bar{\alpha}_0 = \alpha_0 - 1, \qquad \bar{\alpha}_1 = \alpha_1 + 1,$$
$$\bar{q} + q = -\alpha_1 p^{-1}, \qquad \bar{p} + p = t + 2\bar{q}^2. \qquad \text{d-P}(E_7)$$

Remark. One may consider a non-commutative analog for the continuous limit as follows. For the simplicity, suppose that we have a difference equation for the functions f_n. One can take the change of variables with the commutative parameter ε

$$z = \varepsilon\, n \tag{5}$$

supplemented by the maps

$$f_n = F, \qquad f_{n+k} = F + k\,\varepsilon \dot{F} + \tfrac{1}{2} k^2\, \varepsilon^2 \ddot{F} + O(\varepsilon^3). \tag{6}$$

The latter must be chosen in such a way that the limit $\varepsilon \to 0$ exists.

- By using the formulas

  $$q = 1 + \varepsilon^2\, Q - \tfrac{1}{6}\,\varepsilon^3\, P, \quad p = -2 + 2\varepsilon^2\, Q + \tfrac{2}{3}\,\varepsilon^3\, P, \quad t = -6 + \tfrac{1}{3}\,\varepsilon^4\, T, \quad \alpha_1 = 4 + \tfrac{2}{3}\,\varepsilon^4\, T,$$

  the d-P(E_7) has the P_1 system in the limit $\varepsilon \to 0$:

  $$\begin{cases} \dot{q} &= p, \\ \dot{p} &= 6q^2 + t. \end{cases} \qquad \mathrm{P}_1$$

The d-P(E_7) system: a Hamiltonian form

$$\bar{\alpha}_0 = \alpha_0 - 1, \qquad \bar{\alpha}_1 = \alpha_1 + 1,$$
$$\bar{q} + q = -\alpha_1 p^{-1}, \qquad \bar{p} + p = t + 2\bar{q}^2. \qquad \text{d-P}(E_7)$$

Remark. Here we use non-commutative partial derivatives introduced in [Kontsevich, 1993]. Let $f = f(q,p) \in \mathcal{R}$. Its non-commutative derivatives $\partial_q f$, $\partial_p f$ are defined by the identity

$$df = \partial_q f\, dq + \partial_p f\, dp, \tag{7}$$

where it is assumed that additional non-commutative symbols dq, dp are moved to the right by cyclic permutations of generators in monomials.

- Similar to [Veselov, 1991], we call a difference discrete system Hamiltonian if there exists an element $H = H(q, \bar{p}) \in \mathcal{R}$ such that the system can be rewritten in the form

$$p = \partial_q H, \qquad \bar{q} = \partial_{\bar{p}} H. \tag{8}$$

- For the d-P(E_7) system, a Hamiltonian is $H = -q\,\bar{p} + tq + \frac{2}{3}q^3 - \bar{\alpha}_1 \ln \bar{p}$, where for the element $\ln f$ we define the right logarithmic derivative by $d_t(\ln f) := f^{-1}\,\dot{f}$. (cf. with [Mase et al., 2020])
- Then, the non-commutative derivatives are

$$\partial_q H = -\bar{p} + 2q^2 + t, \qquad \partial_{\bar{p}} H = -q - \bar{\alpha}_1\,\bar{p}^{-1}, \qquad \partial_t H = q \tag{9}$$

and (8) is equivalent to the d-P(E_7) system.

Affine Weyl groups and NC discrete systems

Affine Weyl groups W

- Let us fix a generalized Cartan matrix $C = (c_{ij})$, where $i, j \in I := \{0, 1, \ldots, n\}$.
- Sets $\Delta = \{\alpha_0, \ldots, \alpha_n\}$, $\Delta^\vee = \{\alpha_0^\vee, \ldots, \alpha_n^\vee\}$ correspond to simple roots and simple co-roots.
- Denote by $Q = Q(C)$ and $Q^\vee = Q^\vee(C)$ the root and co-root lattices.
  The pairing $\langle \cdot, \cdot \rangle : Q \times Q^\vee \to \mathbb{Z}$ is defined by $\langle \alpha_i, \alpha_j^\vee \rangle = c_{ij}$ and $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$.
- Denote by $W = W(C)$ the Weyl group (or the Coxeter group) defined by generators s_i, $i \in I$:

$$W(C) = \langle s_0, s_1, \ldots, s_n \,|\, s_i^2 = 1,\ (s_i\, s_j)^{m_{ij}} = 1 \rangle, \tag{10}$$

  where the exponents are determined by the value of the product $c_{ij}c_{ji}$ as below

$c_{ij}c_{ji}$	0	1	2	3	≥ 4
m_{ij}	2	3	4	6	∞

- These generators act naturally on Q by reflections

$$s_i(\alpha_j) = \alpha_j - \langle \alpha_i, \alpha_j^\vee \rangle\, \alpha_i = \alpha_j - c_{ij}\, \alpha_i. \tag{11}$$

- Each s_i-action on Q induces an automorphism of the field $\mathbb{C}(\alpha)$ of rational functions in α_i. Hence, $\mathbb{C}(\alpha)$ is a left W-module.
- Recall that one of the important properties of the affine Weyl groups is that they have translations, also known as Kac translations. Let W_0 be a finite Weyl group, $\delta = \sum_{i \in I} k_i\, \alpha_i$ be the null root and $V_0 = \{\mu \in V \,|\, \langle \mu, \delta^\vee \rangle = 0\}$. For an element $\mu \in V_0$ such that $\langle \mu, \mu^\vee \rangle \neq 0$ we define a translation element $t_\mu \in W$ by the formula

$$t_\mu = s_{\delta - \mu}\, s_\mu \tag{12}$$

  and suppose that $w\, t_\mu = t_{w(\mu)}\, w$ for any $w \in W$.

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Extended representations of W & NC discrete dynamics

- The Kac translation acts on simple affine roots as follows

$$t_\mu(\alpha) = \alpha - \langle \mu, \alpha \rangle \, \delta = \alpha - \mu_\alpha \delta. \tag{13}$$

- It is known that the affine Weyl group is decomposed into a semi-direct product of translations in the lattice part M and the finite Weyl group W_0 acting on M, i.e. $W = M \rtimes W_0$. The lattice part M acts on $\mathbb{C}(\alpha)$ as a shift operator, thanks to (13).
- Consider the set of elements $f_i \in \mathcal{R}$, $i \in I$, which we will often call *functions* or *variables*.
- We propose an extension of the representation of W on $\mathbb{C}(\alpha)$ to the skew field $\mathbb{C}(\alpha, f)$ of rational functions in α_i and elements f_i, $i \in I$. One needs to specify the action of s_i on f_j in such a way that the automorphisms s_i on $\mathbb{C}(\alpha, f)$ preserve the Weyl group structure.

Remark 1. Such classes of representations arise naturally from Bäcklund transformations of the differential Painlevé equations (e.g., see the review [Grammaticos and Ramani, 2004]).

- For each $\mu \in M$ we define a set of elements $F_{\mu,i}(\alpha, f) \in \mathbb{C}(\alpha, f)$ by

$$t_\mu(f_i) = F_{\mu,i}(\alpha, f). \tag{14}$$

  This set can be considered as a non-commutative discrete dynamical system.

Remark 2. Sometimes it is necessary to work with an extended Weyl group $\widetilde{W}$. A similar description of the discrete dynamics can be given for $\widetilde{W}$ as well.

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NC d-Painlevé equations: from "continuous" to discrete

- Thanks to the paper [Bershtein et al., 2023], we know that matrix Hamiltonian Painlevé systems of all types have Bäcklund transformations forming an affine Weyl group structure.
- We have reconstructed all the generators for the extended affine Weyl groups corresponding to the NC Hamiltonian systems obtained in [Bobrova and Sokolov, 2023a].
- By using the translation operators similar to those presented in the papers [Sakai, 2001] or [Kajiwara et al., 2017], we have obtained a list of non-commutative discrete systems.
- They might be regarded as non-commutative analogs for the d-Painlevé systems.
- Note that they are connected by the degeneration procedure as follows

$$
\begin{tikzcd}
 & \text{d-P}(D_5)' \arrow[r] \arrow[dr] & \text{d-P}(D_6) \arrow[r] \arrow[dr] & \text{d-P}(D_7) \\
\text{d-P}(D_4) \arrow[ur] \arrow[dr] & & \text{d-P}(E_6) \arrow[r] & \text{d-P}(E_7) \\
 & \text{d-P}(D_5) \arrow[ur] & &
\end{tikzcd}
$$

Remark. The d-P(E_6) and d-P(D_6) systems have a continuous limit to the P_2 system and its another non-equivalent version, respectively. The latter is a subcase of the system derived in [Adler and Sokolov, 2021] and labeled by P_2^2.

NC dressing chains and related discrete systems

NC dressing chains

Remark. The commutative dressing chain was introduced in [Veselov and Shabat, 1993]. It is related to the Painlevé equations [Adler, 1994] and arises from a generalisation of the symmetries for the P_4 and P_5 systems [Noumi and Yamada, 2000]. Quantum dressing chain might be found in [Nagoya, 2004]. Here we **do not (!)** assume any relations for the elements f_i.

- Let $j \in {}^{\mathbb{Z}}\!/_{(n+1)\mathbb{Z}}$. Consider the systems for $n = 2l$ and $n = 2l+1$, $l \in \mathbb{Z}_{\geq 0}$, respectively

$$\dot{f}_j = \sum_{1 \leq r \leq l} f_j\, f_{j+2r-1} - \sum_{1 \leq r \leq l} f_{j+2r}\, f_j + \alpha_j; \qquad A_{2l}^{(1)}$$

$$\begin{aligned}\tfrac{1}{2}\, t\, \dot{f}_j = &\sum_{1 \leq r \leq s \leq l} f_j\, f_{j+2r-1}\, f_{j+2s} - \sum_{1 \leq r \leq s \leq l} f_{j+2r}\, f_{j+2s+1}\, f_j \\ &+ \Big(\tfrac{1}{2} - \sum_{1 \leq r \leq l} \alpha_{j+2r}\Big) f_j + \alpha_j \sum_{1 \leq r \leq l} f_{j+2r}.\end{aligned} \qquad A_{2l+1}^{(1)}$$

- We will cal them $A_n^{(1)}$, $n \geq 2$ type systems or *dressing chains in the Noumi-Yamada variables*.
- These systems admit Lax pairs.

Lax pairs

- Let $\Psi = \Psi(\lambda, t) \in \mathrm{Mat}_{n+1}(\mathcal{R})$, $\lambda \in \mathcal{Z}(\mathcal{R})$ satisfy the linear system

$$\begin{cases} \partial_\lambda \Psi(\lambda, t) &= \mathcal{A}(\lambda, t)\, \Psi(\lambda, t), \\ \partial_t \Psi(\lambda, t) &= \mathcal{B}(\lambda, t)\, \Psi(\lambda, t), \end{cases} \tag{15}$$

where matrices $\mathcal{A} = \mathcal{A}(\lambda, t)$ and $\mathcal{B} = \mathcal{B}(\lambda, t)$ belong to $\mathrm{Mat}_{n+1}(\mathcal{R})$ and have the form

$$\mathcal{A}(\lambda) = A_0 + A_{-1}\,\lambda^{-1}, \qquad \mathcal{B}(\lambda) = B_1\,\lambda + B_0. \tag{16}$$

- Consider the matrices expressed in terms of the standard unit matrices $E_{r,s} \in \mathrm{Mat}_{n+1}(\mathbb{C})$ as

$$A_0 = E_{1,n} + f_0\, E_{1,n+1} + E_{2,n+1}, \quad A_{-1} = \sum_{1\le r\le n+1} \beta_r\, E_{r,r} + \sum_{1\le r\le n} f_r\, E_{r+1,r} + \sum_{1\le r\le n-1} E_{r+2,r},$$

$$B_1 = E_{1,n+1}, \quad B_0 = \sum_{1\le r\le n+1} g_r\, E_{r,r} + \sum_{1\le r\le n} E_{r+1,r}.$$

- Let $\alpha_0 = 1 + \beta_{n+1} - \beta_1, \quad \alpha_j = \beta_j - \beta_{j+1}, \quad j \in \mathbb{Z}/(n+1)\mathbb{Z} \setminus \{0\}$.

Theorem. [Bobrova, 2024] There exists a set of the g-functions such that the compatibility condition of system (15) is equivalent to either the $A_{2l}^{(1)}$ or $A_{2l+1}^{(1)}$ system.

- For the $A_{2l}^{(1)}$, we have $g_j = -\sum_{1\le r\le l} f_{j+2r}$, where indexes belong to $\mathbb{Z}/(n+1)\mathbb{Z}$.

Lax pairs

- Let $\Psi = \Psi(\lambda, t) \in \mathrm{Mat}_{n+1}(\mathcal{R})$, $\lambda \in \mathcal{Z}(\mathcal{R})$ satisfy the linear system

$$\begin{cases} \partial_\lambda \Psi(\lambda, t) &= \mathcal{A}(\lambda, t)\,\Psi(\lambda, t), \\ \partial_t \Psi(\lambda, t) &= \mathcal{B}(\lambda, t)\,\Psi(\lambda, t), \end{cases} \tag{15}$$

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Bäcklund transformations and NC discrete dynamics

- Let the Cartan matrix C be of type $A_n^{(1)}$, $n \geq 2$ and $I = \{0, 1, \dots, n\}$.
- Let us set

$$s_i(\alpha_i) = -\alpha_i, \qquad s_i(\alpha_j) = \alpha_j + \alpha_i \quad (j = i \pm 1), \qquad s_i(\alpha_j) = \alpha_j \quad (j \neq i \pm 1),$$
$$s_i(f_i) = f_i, \qquad s_i(f_j) = f_j \pm \alpha_i\, f_i^{-1} \quad (j = i \pm 1), \qquad s_i(f_j) = f_j \quad (j \neq i \pm 1),$$
$$\pi(\alpha_j) = \alpha_{j+1}, \qquad \pi(f_j) = f_{j+1}, \qquad j \in \mathbb{Z}/_{(n+1)\,\mathbb{Z}}.$$

Theorem. [Bobrova, 2024] Transformations given above are Bäcklund transformations of the $A_{2l}^{(1)}$ and $A_{2l+1}^{(1)}$ systems. Moreover, they define a birational representation of the extended affine Weyl group of type $A_n^{(1)}$, $n \geq 2$.

- Note that the shift operators are given by

$$T_1 = \pi\, s_n\, s_{n-1} \dots s_1, \qquad T_2 = s_1\, \pi\, s_n \dots s_2, \qquad \dots, \qquad T_{n+1} = s_n \dots s_1\, \pi. \tag{17}$$

- They satisfy the relation $T_1\ T_2\ \dots\ T_{n+1} = 1$.
- Thus, any n of them form a basis for the lattice and we can define a discrete system.

Remark. Cases $n = 2$ and $n = 3$ correspond to the P_4 and P_5 equations and discrete systems labeled by d-P(E_6) and d-P(D_5) respectively. For $n = 1$ one needs to consider the P_2 system.

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Further questions

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A study of the NC d-Painlevé equations

- We expect that these equations admit Lax pairs and have a Hamiltonian form.
- We also expect that they have a continuous limit to known NC "differential" Painlevé systems obtained in [Bobrova and Sokolov, 2023b].
- Commutative d- and differential Painlevé equations are related to the orthogonal polynomials [Van Assche, 2022]. Orthogonal polynomials have a non-commutative analog (see [Gelfand et al., 1995]). We assume that our equations are connected with them.

Other discrete Painlevé equations

- Our method can be applied to the q-discrete Painlevé equations.
- In particular, one may define a NC version of the q-P_6 equation which generalizes the matrix equation obtained in [Kawakami, 2020] to the purely non-commutative case. (an ongoing project)
- What about a non-commutative *ell*-discrete Painlevé equation? (see also [Okounkov and Rains, 2015])

NC geometry related to Painlevé equations

- What is the Okamoto space of initial data of NC "differential" Painlevé equations?
- We would like to generalize the method of Painlevé equations' classification introduced in Sakai's paper [Sakai, 2001]. Recent developments might be found in [Rains, 2019].

Cluster algebras and discrete Painlevé equations

- It is known that discrete Painlevé equations are connected with cluster algebras (see, e.g. [Bershtein et al., 2018]). Might we have the same connection in a NC case?

Many thanks!

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