Symmetric Pairs and Symmetric Spaces.

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§1. Algebraic groups and quantum groups.

§2. Symmetric pairs and quantum symmetric pairs.

§1. Algebraic groups and quantum groups.

§1.1 Chevalley group schemes.

§1.2 Diagonal symmetric spaces.

§1.1. Chevalley group schemes.

- $G_{\mathbb{C}}$ connected reductive group $/\mathbb{C}$. (or any $k=\bar{k}$).
- $T \subset B$, $X = \mathrm{Hom}(T, \mathbb{C}^*)$, $Y = \mathrm{Hom}(\mathbb{C}^*, T)$.
- I : the set of simple roots.

Example: $SL_2(\mathbb{C})$, $B = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$, $T = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$.

- Chevalley (1955) constructed a Chevalley group scheme $G_{\mathbb{Z}}$ associated to the root datum.
- Kostant's $\mathbb{Z}$-form (1966) $\rightsquigarrow$ another construction of $G_{\mathbb{Z}}$.
- Lusztig (2009) gives another construction of $G_{\mathbb{Z}}$ using his theory of canonical bases. $\longleftarrow$ we shall recall.

Example: $SL_{2,\mathbb{Z}} \cong \mathrm{Spec}\left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] / \det = 1 \right).$

we will construct this

$SL_{2,\mathbb{Z}}$: Rings $\longrightarrow$ Groups.

$$R \longmapsto SL_2(R) = \mathrm{Hom}\left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] / \det = 1 , R \right)$$

we will construct this

$G_{\mathbb{C}} \leftrightsquigarrow$ root datum $(X, Y, I, \langle , \rangle, \cdots)$

$\rightsquigarrow$ universal enveloping alg $U(\mathfrak{g})$ over $\mathbb{C}$: $\langle e_i, f_i, h_i \rangle / \sim$

quantization

$\rightsquigarrow$ quantum group $\underbrace{U_q(\mathfrak{g})}^{= U_q}$ over $\mathbb{C}(q)$: $\langle E_i, F_i, K_\mu \rangle / \sim$.

- U_q is a non-cocommutative Hopf algebra.

$(\Delta : U_q \to U_q \otimes U_q)$.

Example: • $G = SL_2(\mathbb{C})$. $\quad U_q = \langle E, F, K_n \rangle \quad n \in \mathbb{Z}$.

$$K_n \cdot K_m = K_{n+m}, \quad K_n \cdot E = q^{2n} E K_n,$$

$$EF - FE = \frac{K_1 - K_{-1}}{q - q^{-1}}. \quad K_n F = q^{-2n} F K_n.$$

• $G = PGL_2(\mathbb{C})$ $\quad U_q = \langle E, F, K_n \rangle \quad n \in \mathbb{Z}$.

$$K_n E = q^{n} E K_n. \quad \cdots$$

Quantum groups recover the classical theory with more information

$$\begin{tikzcd}
\text{enveloping alg } U(\mathfrak{g}) & & \text{quantum group } U_q(\mathfrak{g}) \arrow[ll, dashed, "``q=1"" above] \\
\text{Kostant's } \mathbb{Z}\text{-form } {}_{\mathbb{Z}}U(\mathfrak{g}) \arrow[u, hook] \arrow[d, "\otimes_{\mathbb{Z}} k"] & & \text{Lusztig's } \mathcal{A}\text{-form } {}_{\mathcal{A}}U_q \arrow[ll, "q=1" above] \arrow[u, hook, "\mathcal{A}=\mathbb{Z}[q,q^{-1}]" right] \arrow[d, "q=\sqrt[p]{1}"] \\
\text{Alg. of distributions. } {}_{k}U(\mathfrak{g}) & & U_q \text{ at roots of } 1. \arrow[ll, "q=1" above]
\end{tikzcd}$$

$\neq$ enveloping alg of char $k \neq 0$.

$$\text{canonical bases.} \xleftarrow{q=1,} \bigstar \text{ canonical bases.}$$

$\dot{\mathcal{U}}_q$: the modified form of $\mathcal{U}_q$ ($1 = \sum_{\lambda \in X} 1_\lambda \in \dot{\mathcal{U}}_q$).

☆ $\dot{\mathcal{B}}$: the canonical basis on $\dot{\mathcal{U}}_q$.

($\dot{\mathcal{U}}_q \longrightarrow L(\lambda) \otimes^{w} L(\mu)$, $u \longmapsto u(\eta \otimes \xi)$, $\dot{\mathcal{B}} \longmapsto$ canonical basis).

${}_{\mathcal{A}}\dot{\mathcal{U}}_q$: $\mathcal{A}$-subalg generated by $E_i^{(n)} 1_\lambda$, $F_i^{(n)} 1_\lambda$,

= the free $\mathcal{A}$-subalg spanned by $\dot{\mathcal{B}}$ ($\mathcal{A} = \mathbb{Z}[q, q^{-1}]$).

${}_{R}\dot{\mathcal{U}}_q = R \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathcal{U}}_q$ for any $\mathcal{A} \to R$. ($\dot{\mathcal{U}}_q = \mathbb{Q}(q)\dot{\mathcal{U}}_q$; Kostant's $\mathbb{Z}$-form ; ......)

$q \mapsto ?$

Let $A \to \mathbb{Z}$, $q \mapsto 1$. Consider ${}_{\mathbb{Z}}\dot{U}_q \supset \dot{B}$

Def: $\mathcal{O}_{\mathbb{Z}} \subset \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{U}_q, \mathbb{Z})$ ↙ too large is spanned by dual CB, that is, $\{$ constructed via quantization

$b^* \ (b \in \dot{B})$ such that $b^*(b') = \delta_{b,b'}$.

$\boxed{\text{Thm (Lusztig 2009)}}$

- $\mathcal{O}_{\mathbb{Z}}$ (and $\mathcal{O}_{\Bbbk}$) is a commutative Hopf alg. and an integral domain.
- $\mathrm{Spec}\, \mathcal{O}_{\mathbb{Z}} \cong G_{\mathbb{Z}}$, $\mathcal{O}_{\Bbbk}$ $= \mathcal{O}_{\Bbbk} = \mathcal{O}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. is the coordinate ring of $G_{\Bbbk}$ $(\Bbbk = \bar{\Bbbk})$
- $G_{\mathbb{Z}}(R) = \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathcal{O}_{\mathbb{Z}}, R) \longrightarrow {}_R\hat{\dot{U}}_q$, $\phi \longmapsto \sum_{b \in \dot{B}} \phi(b^*)\, b$ ← group like element.

Remarks:

- $$\begin{tikzcd} \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \times \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_{\mathfrak{z}}, \mathbb{Z}) \\ \mathcal{O}_{\mathbb{Z}} \times \mathcal{O}_{\mathbb{Z}} \arrow[u, hook] \arrow[r, "\star"] & \mathcal{O}_{\mathbb{Z}} \arrow[u, hook] \end{tikzcd}$$

dual to the stability condition of CB of $\dot{\mathcal{U}}_{\mathfrak{z}}$

§1.2 Diagonal symmetric spaces. ($\Bbbk = \bar{\Bbbk}$).

$G_\Bbbk \cong G_\Bbbk \times G_\Bbbk / \Delta(G_\Bbbk)$ is a symmetric space.

The $G_\Bbbk \times G_\Bbbk$-mod $\Bbbk[G_\Bbbk]$ is multiplicity free.

($B_\Bbbk \times B_\Bbbk$-eigenspaces are 1-dim).

If char $\Bbbk = 0$, then $\Bbbk[G_\Bbbk] = \bigoplus_{\lambda \in X^+} V^*(\lambda) \otimes V^*(-w_0\lambda)$

If char $\Bbbk > 0$, then $\Bbbk[G_\Bbbk] = \bigcup_{\lambda \in X^+} \Bbbk[G_\Bbbk]_{\leq \lambda}$. good filtration

such that $\Bbbk[G_\Bbbk]_{\leq \lambda} / \Bbbk[G_\Bbbk]_{< \lambda} \cong V^*(\lambda) \otimes V^*(-w_0\lambda)$.

constructed using U_q.

The Chevalley group schemes give the $\mathbb{Z}$-model for $G_{\Bbbk} \cong \dfrac{G_{\Bbbk} \times G_{\Bbbk}}{\Delta(G_{\Bbbk})}$. (Recall $G_{\mathbb{Z}} \times_{\mathbb{Z}} \Bbbk \cong G_{\Bbbk}$).

We next construct the good filtration on $\Bbbk[G_{\Bbbk}]$ using $U_q(\mathfrak{g})$. (It is crucial to work over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$).

Recall

$\dot{\mathcal{U}}_q$: the modified form of $\mathcal{U}_q$ ($1 = \sum_{\lambda \in X} 1_\lambda \in \dot{\mathcal{U}}_q$).

☆ $\dot{\mathcal{B}}$: the canonical basis on $\dot{\mathcal{U}}_q$.

($\dot{\mathcal{U}}_q \longrightarrow L(\lambda) \otimes^w L(\mu)$, $u \longmapsto u(\eta \otimes \xi)$, $\dot{\mathcal{B}} \longmapsto$ canonical basis).

${}_{\mathcal{A}}\dot{\mathcal{U}}_q$: the free $\mathcal{A}$-subalg spanned by $\dot{\mathcal{B}}$ ($\mathcal{A} = \mathbb{Z}[q, q^{-1}]$).

$\mathcal{O}_{\mathbb{Z}} \subset \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathcal{U}}_q, \mathbb{Z})$ spanned by dual CB .

Cell ideals

${}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda]$: ideals in ${}_{\mathcal{A}}\dot{U}_q = \bigcup_{\lambda\in X^+} {}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda]$

$\parallel$

$\{x \in {}_{\mathcal{A}}\dot{U}_q \mid x|_{V(\lambda')} \neq 0 \iff \lambda' \not\leq \lambda\}$.

☆ $\dot{B}[\not\leq\lambda] := \dot{B} \cap {}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda]$ the canonical basis of ${}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda]$

$\dot{B}[\leq\lambda] = \dot{B} - \dot{B}[\not\leq\lambda]$ the image of CB of ${}_{\mathcal{A}}\dot{U}_q[\leq\lambda] := {}_{\mathcal{A}}\dot{U}_q / {}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda]$

$0 \longrightarrow {}_{\mathcal{A}}\dot{U}_q[\not\leq\lambda] \longrightarrow {}_{\mathcal{A}}\dot{U}_q \longrightarrow {}_{\mathcal{A}}\dot{U}_q[\leq\lambda] \longrightarrow 0$ is based.

CB $\longmapsto$ CB or $\{0\}$.

$${}_{\mathcal{A}}\dot{U}_q[\lambda]$$

splits over $\mathbb{C}(q)$

$$0 \longrightarrow {}_{\mathcal{A}}\dot{U}_q[<\lambda] \longrightarrow {}_{\mathcal{A}}\dot{U}_q[\leq\lambda] \longrightarrow {}_{\mathcal{A}}V(\lambda) \otimes_{\mathcal{A}} {}_{\mathcal{A}}V(-w_0\lambda) \longrightarrow 0 \quad \text{[Lusztig]}$$

for any $\mathcal{A} \to \Bbbk$ (not necessarily flat). ☆☆☆

splits over $\mathbb{C}$.

$$0 \longrightarrow {}_{\Bbbk}\dot{U}_q[<\lambda] \longrightarrow {}_{\Bbbk}\dot{U}_q[\leq\lambda] \longrightarrow {}_{\Bbbk}V(\lambda) \otimes_{\Bbbk} {}_{\Bbbk}V(-w_0\lambda) \longrightarrow 0$$

non-example: $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$

$$\mathcal{O}_{\mathcal{A}} = \bigcup_{\lambda \in X^+} \mathrm{Hom}_{\mathcal{A}}\big(\overset{\text{f.d.}}{{}_{\mathcal{A}}\dot{U}_q[\le \lambda]}, \mathcal{A}\big).$$

$\Bbbk[G_\Bbbk] \cong \mathcal{O}_\Bbbk \cong \mathcal{O}_\mathcal{A} \otimes_\mathcal{A} \Bbbk$. has a <u>good filtration</u>.

where $\Bbbk[G_\Bbbk]_{\le \lambda} = \mathrm{Hom}_\mathcal{A}({}_\mathcal{A}\dot{U}_q[\le \lambda], \mathcal{A}) \otimes_\mathcal{A} \Bbbk$ is spanned by the dual CB in $B[\le \lambda]$.

§2. Symmetric pairs and quantum symmetric pairs.

§2.1 Symmetric subgroup schemes.

§2.2 Symmetric spaces

§2.1 Symmetric subgroup schemes. (char $\Bbbk \neq 2$).

$\theta : G_{\Bbbk} \to G_{\Bbbk}$ be an involution. (The classification is independent of $\Bbbk$ by [Springer])

Choose $B_{\Bbbk} \subset G_{\Bbbk}$ such that $B_{\Bbbk} \xrightarrow{w_0 w_0} \theta(B_{\Bbbk})$ is maximal.

Let $K_{\Bbbk} = G_{\Bbbk}^{\theta}$. ($K_{\Bbbk} \neq K_{\Bbbk}^{\circ}$ in general).

Then $K_{\Bbbk} \cdot B_{\Bbbk}$ is the unique open orbit on $G_{\Bbbk}/B_{\Bbbk}$:

Satake diagram.

- θ induces $\tau: I \longrightarrow I$ and $I = I_\circ \sqcup I_\bullet$
- $\theta: X \to X$, $\theta: Y \to Y$, $X_\imath = X / X^\theta$, $Y^\imath = Y^\theta$.

$(G_k, \theta) \longleftrightarrow$ $\imath$-root datum $\rightsquigarrow$ quantum symmetric pairs.

$(U_q, U_q^\imath)$

(Recall root datum $\rightsquigarrow$ quantum groups).

Example :

$G_{\mathbb{C}} = SL_{2,\mathbb{C}}$. $\theta : SL_2(\mathbb{C}) \to SL_2(\mathbb{C})$; $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}$

$\theta' : SL_2(\mathbb{C}) \longrightarrow SL_2(\mathbb{C})$. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} A \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} .$$

$G_{\mathbb{C}}^{\theta} \cong G_{\mathbb{C}}^{\theta'}$ over $\mathbb{C}$.

$\imath$-quantum group ($\imath$ = invariant, involution)

Def: $U_q^\imath \subset U_q$ is generated by

$$B_i = F_i + \varsigma_i T_{w_\bullet}(E_{\tau i})\tilde{K}_i^{-1} \quad (i \in I_\circ), \quad K_\mu \quad (\mu \in Y^\imath), \ldots$$

coideal subalg: $\Delta : U_q^\imath \longrightarrow U_q^\imath \otimes U_q$.

- QSP was originally defined by Letzter
- In [B-Wang], we developed the theory of $\boxed{\text{canonical bases}}$

$\rightsquigarrow$ reflection equation, twisted Yangians, categorifications, Hall algebras, ...

$\dot{U}^\imath_q$: the modified form of $U^\imath_q$ ($1 = \sum_{\lambda \in X_\imath} 1_\lambda \in \dot{U}_q$).

☆ $\dot{B}^\imath$: the $\imath$-canonical basis on $\dot{U}^\imath_q$.

(No satisfactory crystal theory, some progress by [Watanabe]).

${}_{A}\dot{U}^\imath_q$: A-subalg generated by $B^{(m)}_{\bar{\imath},\lambda}$. ($\imath$divided powers).

$=$ the free A-subalg spanned by $\dot{B}^\imath$ ($A = \mathbb{Z}[q, q^{-1}]$).

${}_{R}\dot{U}^\imath_q = R \otimes_A {}_{A}\dot{U}^\imath_q$ for any $A \to R$.

$q \mapsto ?$

Let ${}_{\mathbb{Z}}\dot{\mathrm{U}}^{\imath}_{q} = \mathbb{Z} \otimes_{\mathcal{A}} {}_{\mathcal{A}}\dot{\mathrm{U}}^{\imath}_{q}$ $\quad q \mapsto 1$. (coideal $\rightsquigarrow$ Hopf alg).

Define $\mathcal{O}^{\imath}_{\mathbb{Z}} \subset \mathrm{Hom}_{\mathbb{Z}}({}_{\mathbb{Z}}\dot{\mathrm{U}}^{\imath}_{q}, \mathbb{Z})$ spanned by dual $\imath$CB.

$\boxed{\text{Thm (B-Song 2022)}}$

- $\mathcal{O}^{\imath}_{\mathbb{Z}}$ is a commutative Hopf algebra. (may not be integral)
- $\mathcal{O}_{\mathbb{Z}} \twoheadrightarrow \mathcal{O}^{\imath}_{\mathbb{Z}}$. So. $G^{\imath}_{\mathbb{Z}} := \mathrm{Spec}\, \mathcal{O}^{\imath}_{\mathbb{Z}}$ defines a closed

affine subgroup scheme of $G_{\mathbb{Z}}$. called symmetric subgroup scheme.

$\boxed{\text{Thm (B-Song 2022) continued.}}$

- $\mathcal{O}^2_A = \mathcal{O}^2_{\mathbb{Z}} \otimes_{\mathbb{Z}} A$ is reduced for any domain A with char $\neq 2$.
- The geometric fibers of $G^2_{\mathbb{Z}}$ at k. with char $\neq 2$. are the symmetric subgroup $K_k \subset G_k$ (functorially).
- $\mathcal{O}^2_k$ is the coordinate ring of K_k. ($k = \bar{k}$, char $k \neq 2$).

☆ • Closure of K-orbits on G/B can be defined over $U \subset \mathrm{Sp}\, \mathbb{Z}[\frac{1}{2}]$.

(char $p \rightsquigarrow$ char 0) (conj: $=$)

Remark:

- $\mathcal{O}_{\mathbb{Z}}^2 \times \mathcal{O}_{\mathbb{Z}}^2 \longrightarrow \mathcal{O}_{\mathbb{Z}}^2$ depends the stability of iCB.

(Conj by [B-Wang], proved by [Watanabe]).

- $\mathcal{O}_{\mathbb{Z}} \twoheadrightarrow \mathcal{O}_{\mathbb{Z}}^2$ follows from compatibility of iCB and CB.

- we expect $G_{\mathbb{Z}}^2$ is the fixed point subscheme of $G_{\mathbb{Z}}$.

- [B-Song] we show normality and cohomological vanishing for K_k-orbit closures on G_k/B_k by Frobenius splittings

Example :

$$G_{\mathbb{Z}} = SL_{2,\mathbb{Z}}, \qquad \theta : SL_2(R) \to SL_2(R) : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}$$

$$G^{i}_{\mathbb{Z}}(R) = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| \ a^2 - b^2 = 1,\ a, b \in R \right\}.$$

$$G^{i}_{\mathbb{Z}} = \operatorname{Spec}\left(\mathbb{Z}[x_{11}, x_{12}, x_{21}, x_{22}] \big/ \det = 1,\ x_{11} = x_{22},\ x_{12} = x_{21} \right).$$

$$= \operatorname{Spec}\left(\mathbb{Z}[a, b] \big/ a^2 - b^2 = 1 \right).$$ (not reduced in char 2).

§2.2 Symmetric spaces ($\Bbbk = \overline{\Bbbk}$ of char $\neq 2$)

$K_{\Bbbk} = G_{\Bbbk}^{\theta}$ is reductive by [Steinberg]

$G_{\Bbbk}/K_{\Bbbk}$ is affine and $\Bbbk[G_{\Bbbk}/K_{\Bbbk}] = \Bbbk[G_{\Bbbk}]^{K_{\Bbbk}}$.

$G_{\Bbbk}/K_{\Bbbk}$ is spherical ($B_{\Bbbk} \cdot K_{\Bbbk} / K_{\Bbbk}$ is open)

$\Rightarrow \Bbbk[G_{\Bbbk}/K_{\Bbbk}]$ is multiplicity free.

If char $k = 0$, then.

$$\mathbb{C}[G_{\mathbb{C}}/K_{\mathbb{C}}] \cong \bigoplus_{\lambda \in \check{X}^+} V^*(\lambda).$$

$$\check{X}^+ = X^+ \cap \pi^{-1}(0).$$

$$\pi : X \longrightarrow X_2.$$

If char $k > 0$, ???

$\boxed{\text{Thm (B-Song 2024).}}$

- We define an affine $\mathbb{Z}$-scheme $G/K = \operatorname{Spec} \mathcal{O}[G/K]$.

such that $G/K \times_{\mathbb{Z}} \Bbbk \cong G_{\Bbbk}/K_{\Bbbk}$ for any $\Bbbk = \overline{\Bbbk}$ of char $\neq 2$.

(and $\Bbbk[G_{\Bbbk}/K_{\Bbbk}] = \mathcal{O}[G/K] \otimes_{\mathbb{Z}} \Bbbk$)

- $\mathcal{O}[G/K]$ has a (dual) canonical basis.

- $\Bbbk[G_{\Bbbk}/K_{\Bbbk}] \overset{= \mathcal{O}[G/K] \otimes_{\mathbb{Z}} \Bbbk .}{}$ has a good filtration as a $G_{\Bbbk}$-mod.

Construction of $\mathcal{O}(G/K)$:

$({}_{\mathcal{A}}\dot{U}^{i} \hookrightarrow {}_{\mathcal{A}}\dot{U})$

study the ${}_{\mathcal{A}}\dot{U}^{\imath,+}$-coinvariants of based (right) ${}_{\mathcal{A}}\dot{U}$-mod ${}_{\mathcal{A}}M$.

$$ {}_{\mathcal{A}}M^{\imath}_{*} := {}_{\mathcal{A}}M \Big/ {}_{\mathcal{A}}M \cdot {}_{\mathcal{A}}\dot{U}^{\imath,+} $$

coinvariant

augmentation ideal.

★ $$ ({}_{\mathcal{A}}\dot{U})^{\imath}_{*} := {}_{\mathcal{A}}\dot{U} \Big/ {}_{\mathcal{A}}\dot{U} \cdot {}_{\mathcal{A}}\dot{U}^{\imath,+} . \xrightarrow{\text{dual}} {}_{\Bbbk}\dot{U}^{\imath}\text{-invariant of } \Bbbk[G_{\Bbbk}] . $$

$\|$

$K_{\Bbbk}$-invariant

symmetric subgroup scheme.

Recall.

splits over $\mathbb{C}(q)$ $\quad$ ${}_{\mathcal{A}}\dot{U}_q[\lambda]$

$$0 \longrightarrow {}_{\mathcal{A}}\dot{U}_q[<\lambda] \longrightarrow {}_{\mathcal{A}}\dot{U}_q[\leq\lambda] \longrightarrow {}_{\mathcal{A}}V(\lambda) \otimes_{\mathcal{A}} {}_{\mathcal{A}}V(-w_0\lambda) \longrightarrow 0$$

for any $\mathcal{A} \to \Bbbk$ (not necessarily flat). ☆☆☆

splits over $\mathbb{C}$.

$$0 \longrightarrow {}_{\Bbbk}\dot{U}_q[<\lambda] \longrightarrow {}_{\Bbbk}\dot{U}_q[\leq\lambda] \longrightarrow {}_{\Bbbk}V(\lambda) \otimes_{\Bbbk} {}_{\Bbbk}V(-w_0\lambda) \longrightarrow 0$$

non-example: $0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$.

i-CB $\mapsto$ i-CB or $\{0\}$.

$\boxed{\text{Thm (B-Song 2024)}}$: exact & based. for any $\mathcal{A} \to \Bbbk$.

$\Downarrow$ Good Filtration. $\Downarrow$ (dual) CB

augmentation ideal.

${}_{\mathcal{A}}\mathrm{U}^{\imath}$ - coinvariant quotient.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q \cdot {}_{\mathcal{A}}\mathrm{U}^{\imath,+} \arrow[r] \arrow[d] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q \arrow[r] \arrow[d] & ({}_{\mathcal{A}}\dot{\mathrm{U}}_q)^{\imath}_{*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q[\not\leq\lambda] \cdot {}_{\mathcal{A}}\mathrm{U}^{\imath,+} \arrow[r] \arrow[d] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q[\not\leq\lambda] \arrow[r] \arrow[d] & ({}_{\mathcal{A}}\dot{\mathrm{U}}_q[\not\leq\lambda])^{\imath}_{*} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q[\leq\lambda] \cdot {}_{\mathcal{A}}\dot{\mathrm{U}}^{\imath,+} \arrow[r] \arrow[d] & {}_{\mathcal{A}}\dot{\mathrm{U}}_q[\leq\lambda] \arrow[r] \arrow[d] & ({}_{\mathcal{A}}\dot{\mathrm{U}}_q[\leq\lambda])^{\imath}_{*} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Remark: we "recover"

$$0 \longrightarrow {}_{A}\dot{U}_{\zeta}[<\lambda] \longrightarrow {}_{A}\dot{U}_{\zeta}[\leq\lambda] \xrightarrow{\text{splits over } \mathbb{C}(\zeta)} {}_{A}V(\lambda) \otimes_{A} \underbrace{{}_{A}V(-w_0\lambda)}_{{}_{A}\dot{U}_{\zeta}[\lambda]} \longrightarrow 0$$

by.

for any $A \to k$ (not necessarily flat). ☆☆☆

$$0 \longrightarrow {}_{k}\dot{U}_{\zeta}[<\lambda] \longrightarrow {}_{k}\dot{U}_{\zeta}[\leq\lambda] \longrightarrow {}_{k}V(\lambda) \otimes_{k} {}_{k}V(-w_0\lambda) \longrightarrow 0$$

splits over $\mathbb{C}$.

$$G_k \cong \left. G_k \times G_k \middle/ \Delta(G_k) \right. .$$

easy

hard

too large

Def: $\mathcal{O}(G/K) \subset \mathrm{Hom}_{\mathbb{Z}}(({}_{\mathbb{Z}}\dot{U}^{\imath}_{q})^{*}, \mathbb{Z})$ * is spanned by the dual $\imath$-CB.

based

$\mathcal{O}(G/K)_{\leq\lambda} = \mathrm{Hom}_{\mathbb{Z}}(({}_{\mathbb{Z}}\dot{U}_q[\leq\lambda])^{\imath}_{*}, \mathbb{Z})$ is spanned by the dual $\imath$-CB

$$\mathcal{O}(G/K) = \bigcup_{\lambda\in X^+} \mathcal{O}(G/K)_{\leq\lambda}$$

$$\Bbbk[G_{\Bbbk}/K_{\Bbbk}] = \mathcal{O}(G/K)\otimes_{\mathbb{Z}}\Bbbk.$$

$$\Bbbk[G_{\Bbbk}/K_{\Bbbk}]_{\leq\lambda} = \mathcal{O}(G/K)_{\leq\lambda}\otimes_{\mathbb{Z}}\Bbbk.$$

Remark: * is technically NOT correct, there is NO $\imath$-CB on ${}_{\mathbb{Z}}\dot{U}^{\imath}_{q}$

Good filtration:

$$\mathcal{O}(G/K)_{\le\lambda} \big/ \mathcal{O}(G/K)_{<\lambda} \cong \mathrm{Hom}_{k}\left(({}_{k}\dot{U}_{q}[\le\lambda])^{\wedge}_{*} \big/ ({}_{k}\dot{U}_{q}[<\lambda])^{\wedge}_{*} , k \right)$$

exact $\Longrightarrow$

$$\cong \mathrm{Hom}_{k}\left(\left({}_{k}\dot{U}_{q}[\le\lambda] \big/ {}_{k}\dot{U}_{q}[<\lambda] \right)^{\wedge}_{*} , k \right)$$

$$\cong \mathrm{Hom}_{k}\left(\left({}_{k}V(\lambda) \otimes_{k} {}_{k}V(-w_0\lambda) \right)^{\wedge}_{*} , k \right).$$

$$\cong \mathrm{Hom}_{k}\left({}_{k}V(\lambda) , k \right).$$ or 0.

$$\cong {}_{k}V^{*}(\lambda)$$

Thank you !