Resurgence and instantons in the O(N) sigma model

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- Perturbation theory in quantum field theory
- Borel resummation
- The O(N) nonlinear σ-model in $1+1$ dimension
- Wiener-Hopf solution
- The full trans-series and median resummation

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Successes and problems of perturbation theory in quantum field theory

Anomalous magnetic moment in quantumelectrodynamics

g-factor: magnetic moment of elementary 1/2 spin fermions
(in Bohr-units)

Dirac equation:

$$g = 2$$

quantum field theoretic corrections:

$$a = \frac{g-2}{2}$$

electron:

$$a_e = \sum_{n=1}^{5} c_n^{(e)} \left(\frac{\alpha}{\pi}\right)^n$$

$\alpha \sim \frac{1}{137}$ fine structure constant

$$c_1^{(e)} = \frac{1}{2} \qquad c_2^{(e)} = \frac{197}{144} + \frac{\pi^2}{12} - \frac{\pi^2 \ln 2}{2} + \frac{3}{4}\zeta[3]$$

$c_3^{(e)}$: exactly known (generalized zeta-function)

$c_{4,5}^{(e)}$: known numerically

$$a_e^{\text{theor}} = 0.001\ 159\ 652\ 181\ 64(76)$$

T.Aoyama, T.Kinoshita, M.Nio '19

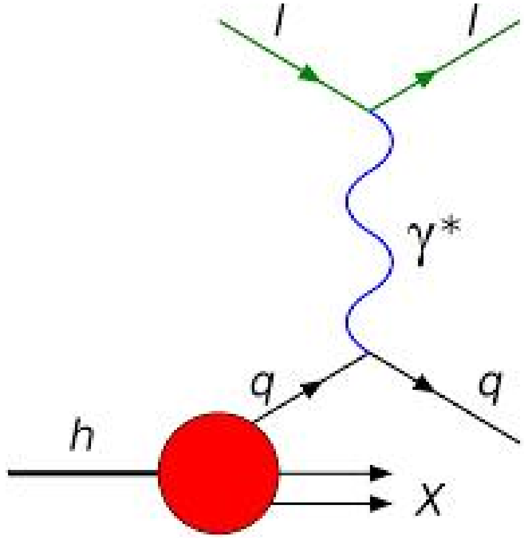
$$a_e^{\text{exp}} = 0.001\ 159\ 652\ 180\ 73(28)$$

agree to 10 digits!

muon: "only" 8-digit agreement

3.7σ deviation [new physics?]

Deep inelastic scattering (QCD)



running coupling:

$$\alpha(E^2) \approx \frac{1}{\beta_o \ln\left(\frac{E^2}{\Lambda^2}\right)} \qquad \text{QCD Lambda – parameter}$$

first 1-2 (3) terms: $E: \ 5 - 80\text{GeV} \qquad \alpha(E^2) \sim 0.15$

Hadronic τ-decay

hadronic branching ratio:

$$R_\tau = A_{\rm EW}\left(|V_{ud}|^2 + |V_{us}|^2\right)\{1+\delta_o\}$$

QCD corrections:

$$\delta_o = \frac{1}{2\pi i}\oint_{|s|=m_\tau^2}\frac{{\rm d}s}{s}\left(1-\frac{s}{m_\tau^2}\right)^3\left(1+\frac{s}{m_\tau}\right)D(s)$$

Adler function:

$$D(s) = \sum_{n=1}^{5} c_n^{(\rm A)}\left(\frac{\alpha(s)}{\pi}\right)^n$$

4 (5) loop order:

$$c_1^{(\mathrm{A})} = 1 \qquad c_2^{(\mathrm{A})} = 1.640 \qquad c_3^{(\mathrm{A})} = 6.371 \qquad c_4^{(\mathrm{A})} = 49.076$$

$$c_5^{(\mathrm{A})} = 277 \pm 51$$

P.A.Baikov, K.G.Chetyrkin, J.H.Kühn '08

Practical problem: $\alpha(s) \sim 0.3$ we know too few terms

Issue of principle!

The perturbative series is asymptotic

$n!$ behaviour

typical perturbative series:

$$f = \sum_{n=0}^{\infty} c_n \alpha^n$$

large n asymptotics:

$$c_n \sim n!$$

- number of Feynman diagrams grows like $n!$
- UV and IR "renormalons" (individual diagramm-series) $\sim n!$

How can we obtain the exact value of the physical quantity knowing the perturbative series f?

Borel resummation

Borel-transform

$$B(t) = \sum_{n=0}^{\infty} \frac{c_n}{n!}\, t^n \qquad \text{convergent}$$

formal Borel summation:

$$\mathcal{S}[f] = \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{-t/\alpha} B(t)$$

$B(t)$ analytic continuation: singularities on the real axis:

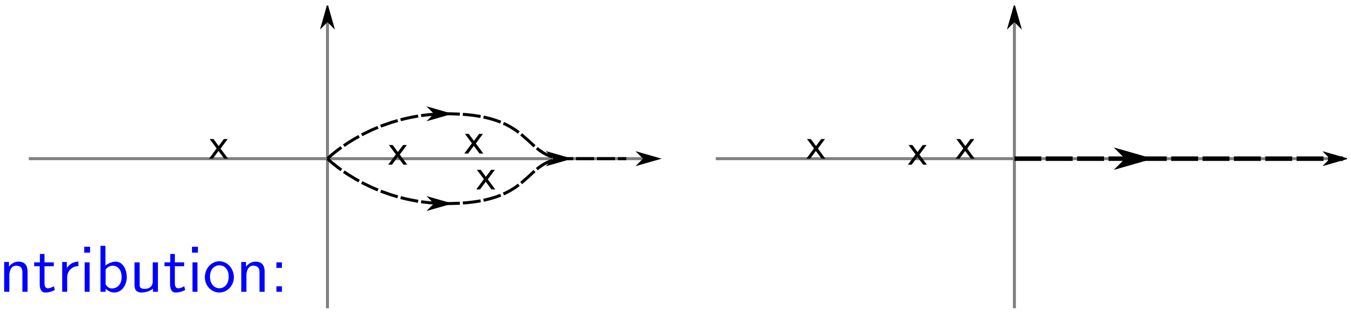
- $t > 0$ IR renormalons and instantons
- $t < 0$ UV renormalons

Lateral Borel-resummation:

$\mathcal{S}_{\pm}[f]$: integration contour slightly above (under) the real axis

- $\mathcal{S}_{+}[f]$ and $\mathcal{S}_{-}[f]$ reproduce the original asymptotic series
- But: $\mathcal{S}_{+}[f] \neq \mathcal{S}_{-}[f]$ NP ambiguity

pole contribution:



$$B(t) \sim \frac{b}{t-S}$$

NP ambiguity:

$$\mathcal{S}_+ - \mathcal{S}_- = 2\pi i b \mathrm{e}^{-S/\alpha}$$

ambiguous imaginary part: $\mathrm{NP} \sim \mathrm{e}^{-S/\alpha}$

'Naive' solution:

$$f_{\mathrm{phys}} = \frac{1}{2}\left(\mathcal{S}_+[f] + \mathcal{S}_-[f]\right) = \mathrm{Re}\,\mathcal{S}_+[f]$$

O(N) model in $1+1$ dimension

O(N) nonlinear σ-model:

$$\mathcal{L} = \frac{1}{2\alpha_o}\partial_\mu \vec{S}\,\partial^\mu \vec{S} \qquad \vec{S}^2 = 1 \qquad \vec{S} = (S_1, \ldots, S_N)$$

- renormalizability $(\alpha_o \to \alpha)$ running coupling
- asymptotic freedom
- dynamical mass-generation (m)
- instantons $(N = 3)$
- integrability: exact scattering matrix known

A.B.Zamolodchikov, Al.B.Zamolodchikov '77

We studied the energy $\epsilon(\rho)$ of the finite ρ density state: $(N = 4)$

$$\epsilon(\rho) = \frac{\pi}{2}\rho^2 \alpha f(\alpha)$$

$$f(\alpha) = \sum_{n=0}^{\infty} \chi_n \, \alpha^n$$

$$\chi_0 = 1 \qquad \chi_1 = \frac{1}{2} \qquad \chi_2 = \frac{1}{4}$$

where

$$\frac{1}{\alpha} - \frac{1}{2}\ln\alpha = \ln\frac{\pi\rho}{\Lambda}$$

TBA equation

$$\chi(\theta) - \int_{-B}^{B} K(\theta - \theta')\chi(\theta')\mathrm{d}\theta' = m\cosh\theta \qquad |\theta| \leq B$$

P.Hasenfratz, M. Maggiore, F. Niedermayer '90

- $K(\theta)$: the logarithmic derivative of the S-matrix
- $\chi(\theta)$: particle density in rapidity space
- B: maximal rapidity of the final density state

$$\rho = \tfrac{1}{2\pi}\int_{-B}^{B}\chi(\theta)\mathrm{d}\theta \qquad \epsilon(\rho) = \tfrac{m}{2\pi}\int_{-B}^{B}\cosh\theta\,\chi(\theta)\mathrm{d}\theta$$

Volin's trick

D.Volin '10

in the $B \to \infty$ limit:

$$\rho = A\frac{\sqrt{B}}{\pi}\hat{\rho} \qquad \epsilon = mA\mathrm{e}^{B}\frac{1}{4\sqrt{2\pi}}\,\hat{\epsilon} \qquad A = m\mathrm{e}^{B}\frac{\sqrt{\pi}}{2\sqrt{2}}$$

$$\hat{\rho} = \sum_{n=0}^{\infty} r_n\, B^{-n} \qquad \hat{\epsilon} = 1 + \sum_{n=0}^{\infty} e_n\, B^{-1-n}$$

- numerical coefficients can be calculated recursively, completely algebraically
- $1/B$ expansion can be rewritten as α-expansion:

$$\frac{1}{\alpha} + \frac{1}{2} - B - \frac{1}{2}\ln B\alpha = \ln\hat{\rho}$$

The first 44 perturbative coefficients analytically

Mathematica: first 44 terms

$$\chi_0 = 1 \qquad \chi_1 = \frac{1}{2} \qquad \chi_2 = \frac{1}{4}$$

$$\chi_3 = \frac{5}{16} - \frac{3}{32}\zeta[3] \qquad \chi_4 = \frac{53}{96} - \frac{9}{64}\zeta[3]$$

$$\vdots \qquad \vdots$$

$$\chi_8 = \frac{30150661}{645120} - \frac{134905}{8192}\zeta[3] + \frac{19719}{8192}\zeta[3]^2 - \frac{145665}{16384}\zeta[5] + \frac{20655}{8192}\zeta[3]\zeta[5] - \frac{496125}{131072}\zeta[7]$$

(unproven) observation: only rationals and odd zetas

Summary of results

Complete trans-series expansion of the energy density:

$$\tilde{\varepsilon} = \sum_{n=0}^{\infty} f_n^{(+)}(v)\,\nu^{kn} \qquad\qquad (k\ N\text{-dependent})$$

$f_0^{(+)} = f_0$ (the original PT series)

NP expansion parameter: $\nu = \mathrm{e}^{-2B}$

running coupling:

$$\frac{1}{v} + \gamma \ln v = 2B \qquad \gamma = \frac{4-N}{N-2} \qquad \nu = v^{-\gamma}\,\mathrm{e}^{-\frac{1}{v}}$$

Using Volin's results we found an algorithm to determine all $f_n^{(+)}(v)$ algebraically

Main conjecture

$$\boxed{\varepsilon_{\rm phys} = \varepsilon_{\rm TBA} = \mathcal{S}_+(\tilde{\varepsilon}) = \sum_{n=0}^{\infty} \nu^{kn} \mathcal{S}_+\big(f_n^{(+)}\big)}$$

no proof

but checked numerically to 72(56) digits for $N = 4(3)$

proved

$$\mathcal{S}_+(\tilde{\varepsilon}) = \mathcal{S}_{\rm med}(f_0) \qquad (N > 3)$$

$\varepsilon_{\rm phys} \neq \mathcal{S}_+(f_0)$ but f_0 "knows" about all NP corrections

(f_0 determines all higher $f_n^{(+)}$)

$N = 3$ $(k = 1)$

f_0, f_1, f_2 not related

$$\varepsilon_{\rm phys} = \varepsilon_{\rm TBA} = \sum_{n=0}^{\infty} \nu^n \mathcal{S}_+\big(f_n^{(+)}\big) = \mathcal{S}_{\rm med}(f_0 + \nu f_1 + \nu^2 f_2)$$

physical interpretation:

f_i: i instanton contribution $i = 0, 1, 2$

TBA(like) integral equation

TBA(-like) integral equation:

$$\chi_n(\theta) - \int_{-B}^{B} \mathrm{d}\theta' K(\theta-\theta')\chi_n(\theta') = \cosh n\theta \qquad |\theta| \le B \qquad n \ge 0$$

TBA context: 2-dim integrable QFT;
$K(\theta)$: logarithmic derivative of S-matrix
$(\mu\cosh\theta, \mu\sinh\theta)$ relativistic 2-momentum
B: (bosonic) fermi-rapidity

densities:

$$O_{n,m} = \frac{1}{2\pi}\int_{-B}^{B} \mathrm{d}\theta \chi_n(\theta)\cosh m\theta \qquad m \ge 0$$

explicit B-dependence dropped:

$$\chi_n(\theta) \sim \chi_n(\theta, B) \qquad O_{n,m} \sim O_{n,m}(B)$$

Relativistic QFT:

$$\varepsilon = \mu^2 O_{1,1} \quad \rho = \mu O_{1,0} \qquad \left\{\varepsilon(B); \rho(B)\right\} \Longrightarrow \varepsilon = \varepsilon(\rho)$$

Differential relations among densities

$$O_{n,m} = O_{m,n} \qquad (I)\ \ \dot{O}_{n,m} = \frac{1}{\pi}\rho_n\rho_m \qquad \dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}B}$$

where

$$\rho_n = \chi_n(B)\ \ [\sim \chi_n(B,B)]$$

$$(II)\ \ \ddot{\rho}_n - n^2\rho_n = \mathcal{F}\rho_n \qquad\qquad \mathcal{F} = \mathcal{F}(B)\ \ \{n\ \text{–independent}\}$$

Ristivojevic '22

1 density determines all:

$$O_{1,1} \longrightarrow \rho_1 \longrightarrow \mathcal{F} \longrightarrow \rho_n \longrightarrow O_{n,m}$$

Wiener-Hopf solution of TBA integral equations

Riemann-Hilbert problem

$$1-\tilde{K}(\omega)=\frac{1}{G_+(\omega)G_-(\omega)} \qquad G_-(\omega)=G_+(-\omega)$$

O(N) NLS model $\quad \{\Delta = 1/(N-2)\}$

$$\tilde{K}(\omega)=\mathrm{e}^{-\pi\Delta|\omega|}\,\frac{\cosh\frac{\pi}{2}(1-2\Delta)\omega}{\cosh\frac{\pi}{2}\omega} \qquad \left\{\tilde{K}(\omega)=\mathrm{e}^{-\pi|\omega|} \quad \mathrm{O}(3)\right\}$$

$\sigma(\omega)=\frac{G_-(\omega)}{G_+(\omega)}$: cut and poles along positive imaginary axis

poles and residues:

$$\sigma(iz\pm\epsilon)\approx \mp i\frac{S_\ell}{z-2\ell\xi_o} \qquad \ell=1,2,\ldots$$

$$\xi_o=\begin{cases}\frac{N-2}{2} & N \text{ even}\\ N-2 & N \text{ odd}\end{cases} \qquad [\xi_o=1 \text{ for O}(3),\ \mathrm{O}(4)]$$

discontinuity:

$$\sigma(iz+\epsilon)-\sigma(iz-\epsilon)=-2i\beta(z)$$

$\beta(z)$: meromorphic with poles at $z=2\ell\xi_o$

expansion in terms of running coupling:

$$2B=\frac{1}{v}+\gamma\ln v+L \qquad \gamma=2\Delta-1$$

$$\mathrm{e}^{-2Bvx}\beta(vx)=\mathrm{e}^{-x}\mathcal{A}(x) \qquad \{\mathcal{A}(x)\sim\mathcal{A}(x,v)\}$$

$$\mathcal{A}(x)=1+\sum_{k=1}^{\infty}(vx)^k\,L_k(\ln x)$$

(L_k polynomial of degree k)

$\nu = \mathrm{e}^{-2B}$: NP expansion parameter $\quad v$: PT expansion parameter

$$c_n = \nu^n \sigma(in + \epsilon)$$

reduced densities:

$$O_{n,m} = \tfrac{1}{4\pi} G_+(in) G_+(im) \mathrm{e}^{(n+m)B} W_{n,m}$$
$$\rho_n = \tfrac{1}{2} G_+(in) \mathrm{e}^{nB} w_n$$

WH integral equation:

$$Q_n(x) + \tfrac{c_n}{n+vx} + i \sum_{\ell=1}^{\infty} \frac{\nu^{2\ell\xi_o} S_\ell q_{n,\ell}}{2\ell\xi_o + vx} + \tfrac{1}{\pi} \int_{\mathcal{C}_+} \frac{\mathrm{e}^{-y} \mathcal{A}(y) Q_n(y)}{x+y} \mathrm{d}y = \frac{1}{n-vx}$$

$$q_{n,\ell} = Q_n\left(\tfrac{2\ell\xi_o}{v}\right)$$

$\mathcal{C}_+$: (just) above the real line densities:

$$w_n = 1 + c_n + i\sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell q_{n,\ell} + \frac{v}{\pi}\int_{\mathcal{C}_+} \mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)\mathrm{d}x$$

conditions: $n, m > 0 \quad n \neq m \quad n, m \neq 2\ell\xi_o$

$$W_{n,m} = \frac{1}{n+m} + \frac{c_m - c_n}{n-m} + c_m k_{n,m} + i\sum_{\ell=1}^{\infty} \frac{\nu^{2\ell\xi_o} S_\ell q_{n,\ell}}{m - 2\ell\xi_o} + \frac{v}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)}{m - vx}\mathrm{d}x$$

$$k_{n,m} = -\frac{c_n}{n+m} - i\sum_{\ell=1}^{\infty} \frac{\nu^{2\ell\xi_o} S_\ell q_{n,\ell}}{m + 2\ell\xi_o} - \frac{v}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-x}\mathcal{A}(x)Q_n(x)}{m + vx}\mathrm{d}x$$

Building blocks: TBA asymptotic series and transseries

Purely perturbative:

$$P_\alpha(x) + \frac{1}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-y}\mathcal{A}(y)P_\alpha(y)}{x+y}\,\mathrm{d}y = \frac{1}{\alpha - vx}$$

well-defined (as asymptotic series) for all $\alpha \neq 0$

$$P_{\alpha,\beta} = \frac{v}{\pi}\int_{\mathcal{C}_+} \frac{\mathrm{e}^{-x}\mathcal{A}(x)P_\alpha(x)}{\beta - vx}\,\mathrm{d}x$$

well-defined (as asymptotic series) for all $\alpha, \beta \neq 0$
perturbatively:

$$W_{\alpha,\beta} \longrightarrow A_{\alpha,\beta} = \frac{1}{\alpha+\beta} + P_{\alpha,\beta}$$

well-defined for all $\alpha, \beta, \alpha+\beta \neq 0$

$$w_\alpha \longrightarrow a_\alpha = 1 + \frac{v}{\pi}\int_{\mathcal{C}_+} \mathrm{e}^{-x}\mathcal{A}(x)P_\alpha(x)\,\mathrm{d}x = \lim_{\beta\to\infty} \beta A_{\alpha,\beta}$$

differential relations for the reduced densities:

$$(n+m)W_{n,m} + \dot{W}_{n,m} = w_n w_m$$
$$\ddot{w}_n + 2n\dot{w}_n = \mathcal{F} w_n$$

perturbative limit:

$$(n+m)A_{n,m} + \dot{A}_{n,m} = a_n a_m$$
$$\ddot{a}_n + 2n\dot{a}_n = \mathcal{F}_o a_n$$

Volin's result determines all:

$$A_{1,1} \longrightarrow a_1 \longrightarrow \mathcal{F}_o \longrightarrow a_n \longrightarrow A_{n,m}$$

Volin's result for $A_{1,1}$:

$$2A_{1,1} = 1 + \frac{v}{2} + \left(\frac{5\gamma}{4} + \frac{9}{8}\right) v^2 + \left(\frac{10\gamma^2}{3} + \frac{53\gamma}{8} + \frac{57}{16}\right) v^3$$
$$+ \frac{v^4}{384}\left(-36\gamma^3(21\zeta_3 - 94) + 10924\gamma^2 + 13344\gamma + 9(144z_3 + 625)\right) + \cdots$$

where

$$z_{2k+1} = 2\frac{\zeta_{2k+1}}{2k+1}(\Delta^{2k+1} - 1 + 2^{-2k-1}) \quad .$$

next step

$$a_1 = 1 + \frac{v}{4} + \left(\frac{5\gamma}{8} + \frac{9}{32}\right) v^2 + \left(\frac{5\gamma^2}{3} + \frac{53\gamma}{32} + \frac{75}{128}\right) v^3$$
$$+ \frac{v^4\left(-288\gamma^3(21\zeta_3-94)+43696\gamma^2+35160\gamma+9(1152z_3+1225)\right)}{6144} + \cdots$$

next step

$$\mathcal{F}_o = -v^2 - 6\gamma v^3 - 26\gamma^2 v^4 + v^5\left(\frac{1}{4}\gamma^3(63\zeta_3 - 386) - 27z_3\right) + \cdots$$

next step

$$a_n = 1 + \frac{v}{4n} + \frac{v^2(20\gamma n+9)}{32n^2} + \frac{v^3\left(640\gamma^2 n^2+636\gamma n+225\right)}{384n^3}$$
$$+ \frac{v^4\left(288n^3\left(\gamma^3(94-21\zeta_3)+36z_3\right)+43696\gamma^2 n^2+35160\gamma n+11025\right)}{6144n^4} + \cdots$$

finally

$$A_{n,m} = \frac{1}{m+n} + \frac{v}{4mn} + \frac{v^2(20\gamma mn+9m+9n)}{32m^2n^2}$$
$$+ \frac{v^3\left(m^2\left(640\gamma^2 n^2+636\gamma n+225\right)+6mn(106\gamma n+39)+225n^2\right)}{384m^3n^3} + \cdots$$

trans-series solution of the linear problem:

$$\tilde{Q}_n(x) = P_n(x) + c_n P_{-n}(x) + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} P_{-2\ell\xi_o}(x)$$

Complete (trans-series) solution:

$$\tilde{q}_{n,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,-2s\xi_o} = A_{n,-2s\xi_o} + c_n A_{-n,-2s\xi_o}$$

$$\tilde{W}_{n,m} = A_{n,m} + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,m} + c_n A_{-n,m} + c_m A_{n,-m}$$
$$+ c_n c_m A_{-n,-m} + i c_m \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{-2\ell\xi_o,-m}$$

$A_{\alpha,\beta}$ (PT) and Stokes constants S_ℓ (NP) universal building blocks!

Energy density $\tilde{W}_{1,1}$

$$n \approx 1 \qquad c_n = \nu\Big\{h_o + \frac{M}{2}(1-n) + \mathrm{O}\big((1-n)^2\big)\Big\} \qquad h_0 = \frac{\delta_{N,3}}{\mathrm{e}}$$

$N \geq 4$ and $N = 3$ cases drastically different!

$$M = \begin{cases} -2\mathrm{e}\left(\frac{\Delta}{\mathrm{e}}\right)^{2\Delta}\frac{\Gamma(1-\Delta)}{\Gamma(1+\Delta)}\mathrm{e}^{i\pi\Delta} & N \geq 4 \\ -\frac{i\pi}{\mathrm{e}} + \frac{2}{\mathrm{e}}(2B + 1 - \gamma_E - \ln 2) & N = 3 \end{cases}$$

$n = 1 \quad m = 1 + \delta \to 1$

$N \geq 4$

$$\tilde{q}_{1,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{1,\ell} A_{-2\ell\xi_o,-2s\xi_o} = A_{1,-2s\xi_o}$$

$$\tilde{W}_{1,1} = A_{1,1} + \tfrac{M\nu}{2} + i \sum_{\ell=1}^{\infty} \nu^{2\ell\xi_o} S_\ell \tilde{q}_{n,\ell} A_{1,-2\ell\xi_o}$$

$N = 3$

$$\tilde{q}_{1,s} - i \sum_{\ell=1}^{\infty} \nu^{2\ell} S_\ell \tilde{q}_{1,\ell} A_{-2\ell,-2s} = A_{1,-2s} + \tfrac{\nu}{\mathrm{e}} A_{-1,-2s}$$

$$\tilde{W}_{1,1} = A_{1,1} + \frac{M\nu}{2} + \frac{2\nu}{\mathrm{e}} P_{1,-1} + \frac{\nu^2}{\mathrm{e}^2} A_{-1,-1}$$
$$+ i \sum_{\ell=1}^{\infty} \nu^{2\ell} S_\ell \tilde{q}_{n,\ell} \left[A_{1,-2\ell} + \tfrac{\nu}{\mathrm{e}} A_{-1,-2\ell} \right]$$

The main conjecture

$N > 3$ case

trans-series:

$$\tilde{\varepsilon} = 2\tilde{W}_{1,1} = 2A_{1,1} + M\nu + 2iS_1 A^2_{1,-k}\nu^k + \mathrm{O}(\nu^{2k}) \qquad k = 2\xi_o$$

$$f_0 = 2A_{1,1}; \;\; f_1^{(+)} = 2iS_1 A^2_{1,-k}; \;\; f_2^{(+)} = 2iS_2 A^2_{1,-2k} - 2S_1^2 A^2_{1,-k} A_{-k,-k};$$

Main conjecture

$$\boxed{\varepsilon_{\mathrm{phys}} = \varepsilon_{\mathrm{TBA}} = \mathcal{S}_+(\tilde{\varepsilon})}$$

no proof but interesting consequences

need new definitions: Stokes automorphism, alien derivatives

Stokes automorphism:

$$\mathfrak{S} = \exp\{-\mathcal{R}\} \qquad \mathcal{R} = \sum_{j=1}^{\infty} \nu^{kj} \Delta_{kj}$$

Δ_{kj} alien derivative at $\omega = kj$

Écalle '81

The definition of alien derivatives are complicated in general, but for singularities closest to the origin in the Borel plane operationally simple:

$f(v) \longrightarrow B[f](t)$ has singularity (cut) at $t = \omega$

$B\big[\Delta_\omega f\big](t) \sim$ discontinuity across the cut

$$\mathcal{S}_-(X) = \mathcal{S}_+(\mathfrak{S}^{-1}X)$$

real X:

$$2i\mathrm{Im}\,\mathcal{S}_+(X) = \mathcal{S}_+\big[(1-\mathfrak{S}^{-1})X\big]$$

Alternatively the alien derivative can be read off the the asymptotic behaviour of the perturbative coefficients:

$$\psi(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n) c_{n-1}}{z^n} \qquad \{z = 2B \text{ for } \mathrm{O}(4)\}$$

$$B[\psi](t) = \sum_{n=0}^{\infty} c_n t^n \qquad \text{(Borel transform)}$$

If asymptotically:

$$c_n = \frac{A_o}{2}\Big\{1 + \frac{p_0}{n} + \frac{p_1}{n(n-1)} + \frac{p_2}{n(n-1)(n-2)} + \dots\Big\}$$
$$+\frac{B_o}{2}(-1)^n\Big\{1 + \frac{q_0}{n} + \frac{q_1}{n(n-1)} + \frac{q_2}{n(n-1)(n-2)} + \dots\Big\}$$

alien derivatives at 1 and -1:

$$(\Delta_1\psi)(z) = -i\pi A_o\Big\{1 + \sum_{m=0}^{\infty} \frac{p_m}{z^{m+1}}\Big\}$$

$$(\Delta_{-1}\psi)(z) = i\pi B_o\Big\{1 - \sum_{m=0}^{\infty} \frac{(-1)^m q_m}{z^{m+1}}\Big\}$$

reality of $\varepsilon_{\rm phys}$:

$$\Delta_{kj} A_{1,1} = 2iS_j A^2_{1,-kj}$$

combining with differential relations:

$$\Delta_{kj} A_{n,m} = 2iS_j A_{n,-kj} A_{m,-kj}$$

complete resurgence structure: all alien derivatives of all building blocks are known

Main result

$$\boxed{\tilde{\varepsilon} = M\nu + \mathfrak{S}^{-1/2} f_0}$$

Median resummation:

$$\mathcal{S}_{\rm med}(X) = \mathcal{S}_+\big(\mathfrak{S}^{-1/2} X\big) = \mathcal{S}_-\big(\mathfrak{S}^{1/2} X\big)$$

Aniceto, Schiappa '13

finally:

$$\varepsilon_{\rm phys} = M\nu + \mathcal{S}_{\rm med}(f_0)$$

f_0 determines full trans-series, “knows” about all higher NP contributions!

$N = 3$

trans-series:

$$\tilde{\varepsilon} = 2\tilde{W}_{1,1} = 2A_{1,1} + M\nu + \frac{4}{\rm e}\nu P_{1,-1} + \frac{2}{{\rm e}^2}\nu^2 A_{-1,-1} + \dots$$

Main conjecture $\longrightarrow$ same resurgence as $N \geq 4$ $(k = 2)$

using Stokes automorphism

$$\tilde{\varepsilon} = \mathfrak{S}^{-1/2}(f_0 + \nu f_1 + \nu^2 f_2)$$

$$f_0 = 2A_{1,1} \qquad f_1 = M_o + \tfrac{4}{\mathrm{e}}P_{1,-1} \qquad f_2 = \tfrac{2}{\mathrm{e}^2}A_{-1,-1}$$

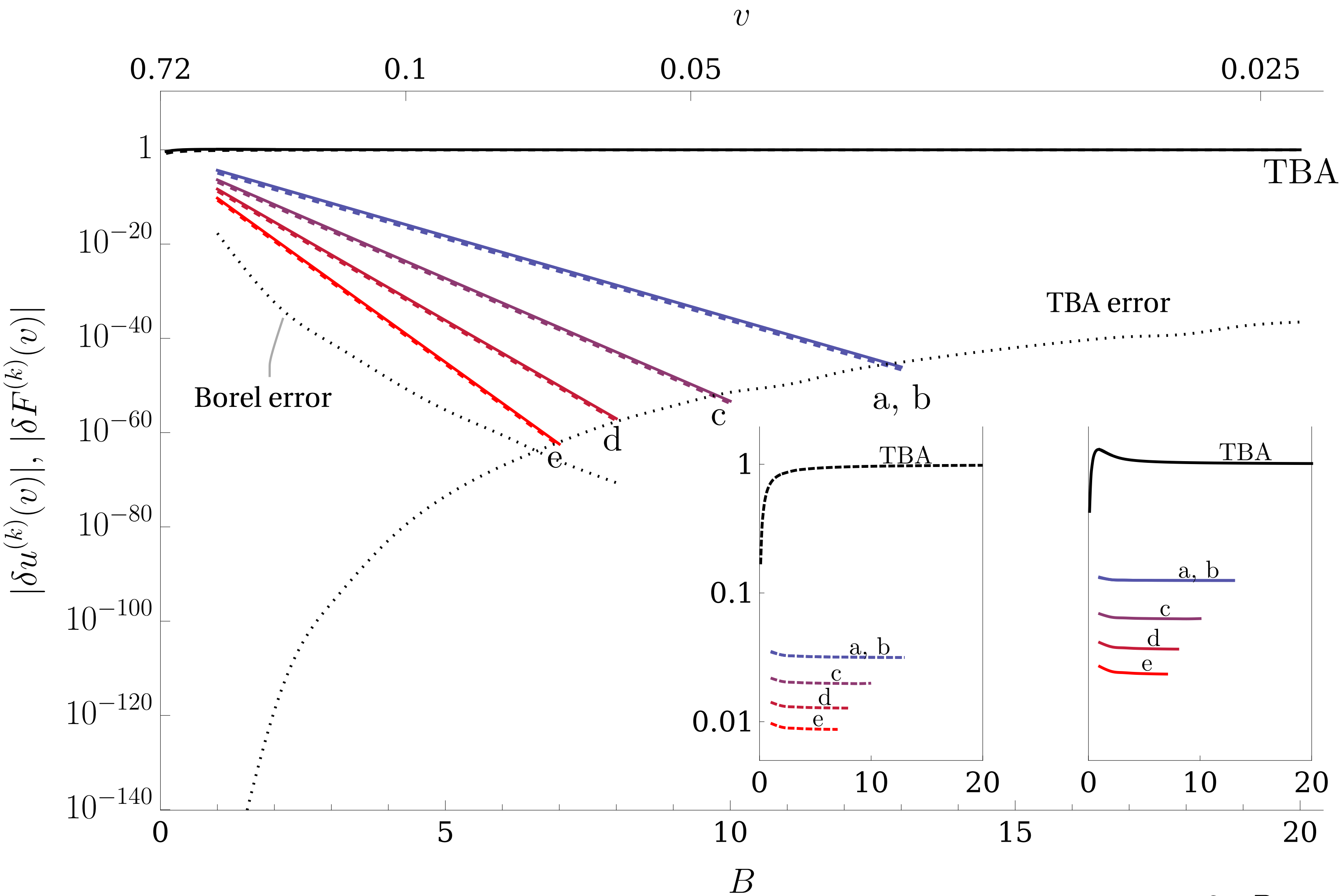
$$M_o = \tfrac{2}{\mathrm{e}}(1 - \gamma_E - \ln 2 + 2B) = \tfrac{2}{\mathrm{e}}\Big(\tfrac{1}{v} + \ln v - \gamma_E - 4\ln 2\Big)$$

median resummation:

$$\varepsilon_{\mathrm{phys}} = \mathcal{S}_{\mathrm{med}}(f_0 + \nu f_1 + \nu^2 f_2)$$

need f_0 (PT) and 1 and 2-instanton sectors

Numerical results



particle and energy density in the $1/B$ expansion, NP orders: e^{-2mB}

conclusion: $$f_{\mathrm{exact}} = f_{\mathrm{TBA}} = S_{\mathrm{med}}(f)$$

Convergence? YES

Thank you!