Bäcklund transformation approach to modified Camassa-Holm equation

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Outline

- Introduction
- Modified Camassa-Holm equation
    * Bäcklund transformation
    * Nonlinear superposition formula
- Application

I. Introduction

In Mathematics,

$$\omega_{\alpha\beta} = \sin\omega \tag{1}$$

likely first appeared in the geometry of pseudospherical surfaces. Here α, β are the parameters of the asymptotic lines, ω is the angle of the asymptotic lines.

Edmond Bour, in a paper for Grand Prix des Mathématiques (1859), first derived this equation. The published version appeared in 1862

Bour, E., 1862b. Théorie de la déformation des surfaces. J. Éc. Imp. Polytech. 22, 1–148.

Upon commenting on the content of this memoir, Liouville wrote: *Every word is an idea. From now on, Mr. Bour has his rank fixed among the masters. He is no longer a promising young man, but a great geometer who has kept the brilliant promises of his youth.*

This equation was rediscovered by Pierre Ossian Bonnet (1867)
Alfred Enneper (1868)

In Physics,

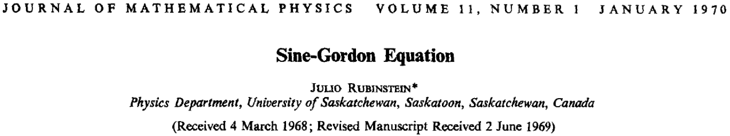
- Crystal dislocation
  In 1953, Seeger, Donth, Kochendörfer recognized the sine-Gordon equation was identical with the equation governing a simple model for dislocations in crystals.
- Relativistic field theory
  Skyrme (1958), Perring - Skyrme (1962)
- Long Josephson Junctions
  Josephson (1965)
- Nonlinear Optics
  Propagation of ultrashort optical pulses
  Lamb (1967)
  ...

Names?

Enneper equation (Seeger suggested)

Nonlinear Klein-Gordon equation (Scott)

The first paper titled sine-Gordon equation:



JOURNAL OF MATHEMATICAL PHYSICS VOLUME 11, NUMBER 1 JANUARY 1970

Sine-Gordon Equation

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(Received 4 March 1968; Revised Manuscript Received 2 June 1969)

sine-Gordon equation (Kruskal suggested)

Bianchi transformation 1879

$$\frac{\partial^2\omega}{\partial u^2} - \frac{\partial^2\omega}{\partial v^2} = \sin\omega\cos\omega$$

Bianchi found

$$\frac{\partial\theta}{\partial u} + \frac{\partial\omega}{\partial v} = \cos\omega\sin\theta, \quad \frac{\partial\theta}{\partial v} + \frac{\partial\omega}{\partial u} = -\sin\omega\cos\theta,$$

Darboux[1] called the above relations the transformation of Bianchi. With the help of the parameters of the asymptotic lines

$$u + v = 2\alpha, \quad u - v = 2\beta$$

the sine-Gordon equation and the Bianchi transformation become

$$\omega_{\alpha\beta} = \sin\omega\cos\omega$$

and

$$(\theta+\omega)_\alpha = \sin(\theta-\omega), \quad (\theta-\omega)_\beta = \sin(\theta+\omega)$$

[1] Darboux, Vol III, p.422

Bäcklund transformation

$$\boxed{\begin{aligned}\sin\sigma\left(\frac{\partial\theta}{\partial u}+\frac{\partial\omega}{\partial v}\right)&=\sin\theta\cos\omega-\cos\sigma\cos\theta\sin\omega,\\ \sin\sigma\left(\frac{\partial\theta}{\partial v}+\frac{\partial\omega}{\partial u}\right)&=-\cos\theta\sin\omega+\cos\sigma\sin\theta\cos\omega.\end{aligned}}$$

The transformation of Bianchi is only a particular case of a transformation discovered by Bäcklund (1883).

$$(\theta+\omega)_\alpha=\frac{1+\cos\sigma}{\sin\sigma}\sin(\theta-\omega),\quad (\theta-\omega)_\beta=\frac{1-\cos\sigma}{\sin\sigma}\sin(\theta+\omega)$$

Bäcklund: Om ytor med konstant negativ krökning, Lund Universitets Arsskrift, Vol. XIX (1883)

Lie transformation

This transformation is immediate when the sine-Gordon equation is reformulated in terms of α, β

$$\omega_{\alpha\beta} = \sin\omega\cos\omega.$$

Now it is evident that if $\omega = \phi(\alpha, \beta)$ be a solution, so also is $\omega_1 = \phi(m\alpha, \beta/m)$.
Lie has called attention to the fact that every Bäcklund transformation is a combination of transformations of Lie and Bianchi.

Permutability Theorem 1892

Bianchi's diagram:

$$\begin{tikzcd}
 & \theta_1 \arrow[dr, "\sigma_2"] & \\
\phi \arrow[ur, "\sigma_1"] \arrow[dr, "\sigma_2"'] & & \omega \\
 & \theta_2 \arrow[ur, "\sigma_1"'] &
\end{tikzcd}$$

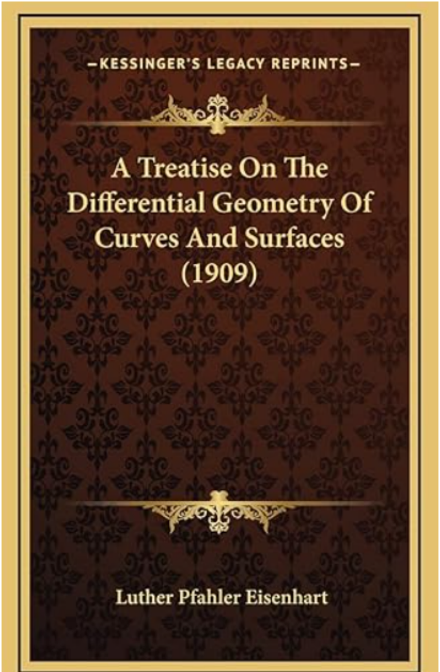
Nonlinear superposition formula:

$$\tan\left(\frac{\phi-\omega}{2}\right)=\frac{\sin\left(\frac{\sigma_1+\sigma_2}{2}\right)}{\sin\left(\frac{\sigma_1-\sigma_2}{2}\right)}\tan\left(\frac{\theta_1-\theta_2}{2}\right).$$

When the transforms of a given pseudospherical surface are known, all the transformations of the former can be effected by algebraic processes and differentiation.

Bianchi, Sulla transformazione di Bäcklund per le superficie psedusferiche. Rend. Lincei 1(1892)

Chapter VIII



—KESSINGER'S LEGACY REPRINTS—

A Treatise On The Differential Geometry Of Curves And Surfaces (1909)

Luther Pfahler Eisenhart

KdV

$$u_t + (6u^2 + u_{xx})_x = 0$$

becomes $w_t = 6w_x^2 - w_{xxx}$ by $u = -w_x$. In terms of w, the 1-soliton solutions are obtained from the completely integrable Pfaffian system

$$w_x = w^2 - k^2, \quad w_t = -4k^2(w^2 - k^2).$$

The set of equations can be modified to obtain a Pfaffian system which will generate a new solution of the Korteweg-de Vries equation from any given solution.

$$\begin{aligned} w'_x &= -w_x - k^2 + (w' - w)^2, \\ w'_t &= -w_t + 4\left[k^2 u' + u^2 - u(w' - w)^2 - u_x(w' - w)\right]. \end{aligned}$$

Superposition principle

$$w_{12} = w + \frac{k_2^2 - k_1^2}{w_2 - w_1}.$$

Wahlquist & Estabrook (1973)

MKdV, NLS

MKdV: $u_t + 6u^2u_x + u_{xxx} = 0$

BT:

$$\begin{aligned} w'_x &= -w_x + 2k\sin(w' - w), \\ w'_t &= -w_t - 8k^2u - 4ku_x\cos(w' - w) - 4(2k^3 + ku^2)\sin(w' - w). \end{aligned}$$

Wadati (1974); Lamb (1974); Hirota (1974); Chen (1974).

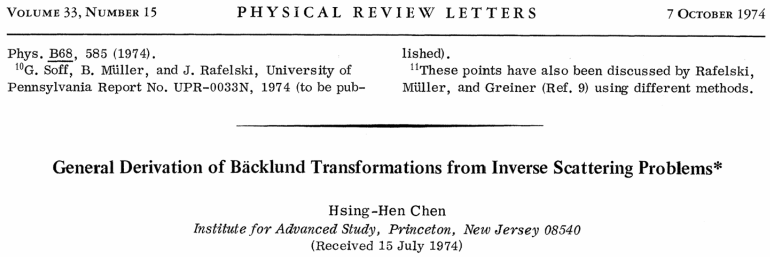
NLS $iq + r + 2|z|^2z = 0.$

BT:

$$\begin{aligned} p &= p' + n, \\ q &= q' + \tau(p + p') - kn + \frac{1}{4}\mathrm{i}(|w|^2 + |v|^2), \end{aligned}$$

where $w = z + z'$, $v = z - z'$, $\tau = \pm\mathrm{i}(b - 2|v|^2)^{1/2}$, $n = -(\mathrm{i}/2)w\tau + \mathrm{i}kv$. $p = \partial z/\partial x, q = \partial z/\partial y, r = \partial^2 z/\partial x^2$.

Lamb (1974) used a method of Clairin (1903); Chen (1974).



VOLUME 33, NUMBER 15 PHYSICAL REVIEW LETTERS 7 OCTOBER 1974

Phys. B68, 585 (1974).

[10] G. Soff, B. Müller, and J. Rafelski, University of Pennsylvania Report No. UPR-0033N, 1974 (to be published).

[11] These points have also been discussed by Rafelski, Müller, and Greiner (Ref. 9) using different methods.

General Derivation of Bäcklund Transformations from Inverse Scattering Problems*

Hsing-Hen Chen
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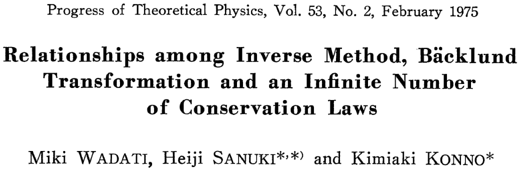
(Received 15 July 1974)



Progress of Theoretical Physics, Vol. 53, No. 2, February 1975

Relationships among Inverse Method, Bäcklund Transformation and an Infinite Number of Conservation Laws

Miki WADATI, Heiji SANUKI*,*) and Kimiaki KONNO*

Darboux to Bäcklund

Consider the Schrödinger equation

$$-\psi_{xx} + \eta^2 \psi = u\psi. \tag{4.1}$$

Let $\psi_0(x)$ be some particular solution of Eq.(4.1) for $\eta = \eta_0$, and $\psi(x, \eta)$ be an arbitrary solution of Eq.(4.1). Define

$$\psi'(x, \eta) = \frac{W\{\psi(x, \eta); \psi_0(x)\}}{(\eta^2 - \eta_0^2)\psi_0(x)},$$

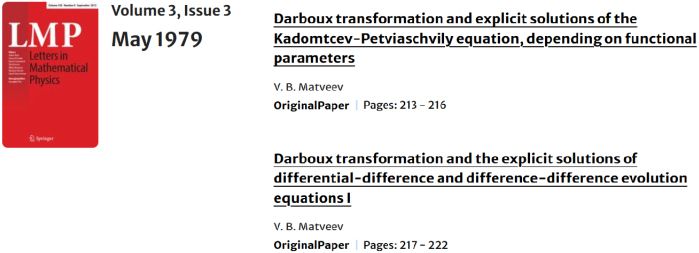
where $W\{\psi; \phi\}$ is the Wronskian of the functions ψ and ϕ.

THEOREM. *The function $\psi'(x, \eta)$ which is defined by Eq.(4·2) is a solution of Eq.(4·1) with a potential $u'(x) = u(x) + \Delta u(x)$, where*

$$\Delta u(x) = 2(\log \psi_0(x))_{xx}.$$

Wadati, Sanuki, Konno (1975) did not mention Darboux. Rather, they referred to Crum (1955).

The term "Darboux transformation" was first introduced by Matveev (1979).



Volume 3, Issue 3

May 1979

Darboux transformation and explicit solutions of the Kadomtcev-Petviaschvily equation, depending on functional parameters

V. B. Matveev

OriginalPaper | Pages: 213 – 216

Darboux transformation and the explicit solutions of differential-difference and difference-difference evolution equations I

V. B. Matveev

OriginalPaper | Pages: 217 – 222

Why Bäcklund transformations

- Integrability
- Construction of solutions
- Discretization
  Levi (1980);
  Hietarinta, Joshi, Nijhoff: Discrete Systems and Integrability (2016)
- Numerical analysis
  ϵ - algorithm Wynn (1956)
- Analysis of PDEs
  Stability of solitons, global existence.
  Mizumachi, Pelinovsky, Shimabukuro (2010)
- ...

II. Modified Camassa-Holm

$$u_t - u_{xxt} + (u_x^2 - u^2)u_{xxx} + (3u^2 - u_x^2 - 4uu_{xx} + 2u_{xx}^2)u_x = 0,$$

or

$$m_t + [m(u^2 - u_x^2)]_x = 0, \;\; m = u - u_{xx}.$$

- Fuchssteiner & Fokas (1981) proposed

  $$\Phi_6(u) = \{\alpha + \beta\partial^2 + \delta\partial u \partial^{-1} u\}(1 - \partial^2)^{-1}$$

  and

  $$u_{t_n} = \Phi_6(u)^n u_x, \;\; n = 0, 1, 2, ...$$

  which includes the mCH equation.

- Fokas (1995) wrote down

$$\begin{aligned} u_t + u_x - \frac{3}{2}\rho_2\beta u_{xxt} + \left(1 - \frac{3}{2}\rho_2\right)\beta u_{xxxx} + \alpha u u_x \\ - \frac{1}{2}\rho_2\alpha\beta\left(u u_{xxx} + 2u_x u_{xx}\right) + 3\mu\alpha^2 u^2 u_x \\ - \frac{3}{2}\mu\rho_2\alpha^2\beta\left(u^2 u_{xxx} + u_x^3 + 4u u_x u_{xx}\right) \\ + \frac{9}{4}\mu\rho_2^2\alpha^2\beta^2\left(u_x^2 u_{xxx} + 2u_x u_{xx}^2\right) = 0 \end{aligned}$$

- Fuchssteiner (1996)
- Olver & Rosenau (1996) gave the explicit form

$$m_t = u_t - u_{xxt} = \frac{1}{2}\left[(u^2 - u_x^2)(u - u_{xx})\right]_x .$$

- Schiff (1996) worked out its Lax representation

$$A_t - B_x + [A, B] = 0$$

  where

$$A = \begin{pmatrix} m/\sqrt{\lambda} & 1 \\ 1 & -m/\sqrt{\lambda} \end{pmatrix},$$

$$B = \begin{pmatrix} \sqrt{\lambda}(m - \frac{1}{4}u_{xx} + ms/\sqrt{\lambda} & \lambda + s + \frac{1}{2}\sqrt{\lambda}u_x \\ \lambda + s - \frac{1}{2}\sqrt{\lambda}u_x & -\sqrt{\lambda}(m - \frac{1}{4}u_{xx} - ms/\sqrt{\lambda} \end{pmatrix}$$

  with $s = \frac{1}{2}\left(\frac{1}{4}m_x^2 - m^2\right)$.
- mCH reappeared in a paper by Qiao (2006).

You may come across another name for this equation: the FORQ equation, a designation adopted by some researchers.

- Hirota Bilinear approach
  Matsuno (2013)
- Stability problems for peakons
  Liu, Liu, Qu (2014)
- Peakons
  CHANG, Szmigielski (2018)
- Large-time asymptotics for the Cauchy problem
  Boutet de Monvel, Karpenko, Shepelsky (2009).
- mCH with nonzero boundary conditions
  Boutet de Monvel, Karpenko, Shepelsky (2020).
- Soliton resolution conjecture
  Yang, Fan (2022).
- ...

associated mCH & its BT

Linear spectral problem or Lax pair

$$\Phi_x = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\lambda m \\ \frac{1}{2}\lambda m & \frac{1}{2} \end{pmatrix} \Phi,$$

and

$$\Phi_t = \begin{pmatrix} \frac{1}{\lambda^2} + \frac{1}{2}(u^2 - u_x^2) & -\frac{1}{\lambda}(u - u_x) - \frac{1}{2}\lambda m(u^2 - u_x^2) \\ \frac{1}{\lambda}(u + u_x) + \frac{1}{2}\lambda m(u^2 - u_x^2) & -\frac{1}{\lambda^2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix} \Phi.$$

We may introduce a coordinate change $(x, t) \to (y, \tau)$ as follows

$$dy = m dx - m(u^2 - u_x^2) dt, \quad d\tau = dt,$$

thus

$$\frac{\partial}{\partial x} = m \frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - m(u^2 - m^2 u_y^2) \frac{\partial}{\partial y}. \tag{2}$$

amCH

Then we have the amCH:

$$m_\tau + 2m^3 u_y = 0, \tag{3a}$$

$$u = m + \frac{1}{2} m \left(\frac{1}{m} \right)_{y\tau}, \tag{3b}$$

and its spectral problem

$$\Phi_y = U\Phi,\ U = \begin{pmatrix} -\frac{1}{2m} & \frac{1}{2}\lambda \\ -\frac{1}{2}\lambda & \frac{1}{2m} \end{pmatrix}, \tag{4}$$

and

$$\Phi_\tau = V\Phi,\ V = \begin{pmatrix} \frac{1}{\lambda^2} & \frac{1}{\lambda}(mu_y - u) \\ \frac{1}{\lambda}(mu_y + u) & -\frac{1}{\lambda^2} \end{pmatrix}. \tag{5}$$

DT/BT for amCH

We consider a gauge transformation $\hat{\Phi} = T\Phi$. That is, T has to satisfy

$$T_y + TU - \hat{U}T = 0, \tag{6a}$$

$$T_\tau + TV - \hat{V}T = 0, \tag{6b}$$

where

$$\hat{U} = U\Big|_{\substack{m\to\hat{m}\\ u\to\hat{u}}}, \quad \hat{V} = V\Big|_{\substack{m\to\hat{m}\\ u\to\hat{u}}}.$$

Take the simplest ansätz

$$T = \lambda F + G, \quad F = (f_{ij})_{2\times 2}, \quad G = (g_{ij})_{2\times 2},$$

then it is found that F and G may take the following forms

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$$

and from (6a) we find that g_{11}, g_{22} satisfy

$$\frac{g_{11,y}}{g_{11}} = \frac{1}{2}\left(\frac{1}{m} - \frac{1}{\hat{m}}\right), \quad \frac{g_{22,y}}{g_{22}} = \frac{1}{2}\left(\frac{1}{\hat{m}} - \frac{1}{m}\right), \quad g_{22} - g_{11} = \frac{1}{m} + \frac{1}{\hat{m}}$$

which, along with $g_{11} = \alpha a,\ g_{22} = -\frac{\alpha}{a}$, imply

$$\boxed{\hat{m} = \frac{ma}{a - 2ma_y}}, \tag{7a}$$

$$\boxed{a_y = \frac{a}{m} + \frac{\alpha}{2}(1 + a^2)}, \tag{7b}$$

where $a = a(y, \tau)$ and α is an arbitrary constant.
From (6b), we have

$$\hat{u} - u = \alpha a_\tau + m u_y + \hat{m}\hat{u}_y, \quad \hat{u} - u = \frac{\alpha a_\tau}{a^2} - m u_y - \hat{m}\hat{u}_y, \tag{8}$$

$$\hat{m}\hat{u}_y - \hat{u} = \frac{2a}{\alpha} - a^2(m u_y - u), \quad \hat{m}\hat{u}_y + \hat{u} = -\frac{2}{\alpha a} - \frac{1}{a^2}(m u_y + u), \tag{9}$$

The first two equations of (8) gives

$$\boxed{\hat{u} = u + \frac{1}{2}\alpha(1 + \frac{1}{a^2})a_\tau} \tag{10}$$

and taking above equation, (7) and (3a) into consideration, (8) holds. Now, substituting (7) into (9) and noting the mCH equation, we obtain

$$\boxed{a_\tau = \frac{1}{\alpha}[mu_y(a^2 - 1) - u(a^2 + 1)] - \frac{2a}{\alpha^2}}. \tag{11}$$

Bäcklund for mCH

According to the reciprocal transformation, BT (7)(10)(11) may be reformulate as

$$\hat{m} = -\frac{m}{1+\alpha m(a+\frac{1}{a})}, \tag{12}$$

$$\hat{u} = \frac{1}{2}(u_x - u)a^2 - \frac{1}{2a^2}(u_x + u) - \frac{a^2+1}{a\alpha}, \tag{13}$$

where

$$a_x = a + \frac{1}{2}\alpha m(a^2+1), \tag{14}$$

$$\begin{aligned} a_t = & \left(\frac{1}{\alpha} + \frac{\alpha m}{2}(u+u_x)\right)(u_x - u)a^2 + \left(u_x^2 - u^2 - \frac{2}{\alpha^2}\right)a \\ & - \left(\frac{1}{\alpha} + \frac{\alpha m}{2}(u - u_x)\right)(u_x + u). \end{aligned} \tag{15}$$

Because of the reciprocal transformation, the BT for the mCH equation must involve the independent variable x. From the reciprocal transformation, we have

$$dx = \frac{1}{m}dy + (u^2 - m^2u_y^2)d\tau, \tag{16}$$

thus

$$d\hat{x} - dx = \left(\frac{1}{\hat{m}} - \frac{1}{m}\right)dy + \left(\hat{u}^2 - u^2 - \hat{m}^2\hat{u}_y^2 + m^2u_y^2\right)d\tau.$$

(7a) implies $\frac{1}{\hat{m}} - \frac{1}{m} = -\frac{2a_y}{a}$, and

$$\hat{u}^2 - u^2 - \hat{m}^2\hat{u}_y^2 + m^2u_y^2 \overset{\text{(10)(7b)(3a)}}{=}$$
$$(ua^2 - ma^2u_y + mu_y + u + \alpha a_\tau)\frac{\alpha a_\tau}{a^2} \overset{\text{(11)}}{=} -\frac{2a_\tau}{a},$$

thus

$$d\hat{x} - dx = -\frac{2a_y dy + 2a_\tau d\tau}{a} = -2d\ln|a|,$$

or

$$\hat{x} - x = -2\ln|a| + C. \tag{17}$$

Finally we have the BT of mCH equation as

$$\begin{aligned}
\hat{u} &= \frac{1}{2}(u_x - u)a^2 - \frac{1}{2a^2}(u_x + u) - \frac{a^2+1}{a\alpha}, \\
\hat{x} &= x - 2\ln|a|, \qquad \hat{t} = t.
\end{aligned}$$

where

$$\begin{aligned}
a_x &= a + \frac{1}{2}\alpha m(a^2+1), \\
a_t &= \left(\frac{1}{\alpha} + \frac{\alpha m}{2}(u + u_x)\right)(u_x - u)a^2 + \left(u_x^2 - u^2 - \frac{2}{\alpha^2}\right)a \\
&\quad - \left(\frac{1}{\alpha} + \frac{\alpha m}{2}(u - u_x)\right)(u_x + u).
\end{aligned}$$

Wang, Mao, L (2020)

NSF

Schematically, we may either

$$\begin{tikzcd}
u \arrow[r, Rightarrow, "\alpha", "a_1"'] & u_1 \arrow[r, Rightarrow, "\beta", "a_{12}"'] & u_{12}, \\
u \arrow[r, Rightarrow, "\beta", "a_2"'] & u_2 \arrow[r, Rightarrow, "\alpha", "a_{21}"'] & u_{21}.
\end{tikzcd}$$

We start with the amCH. From its Bäcklund transformation, we may have

$$T(a_{12},\beta)T(a_1,\alpha)=T(a_{21},\alpha)T(a_2,\beta)$$

which implies

$$a_{12}=\frac{\alpha a_2-\beta a_1}{a_1(\beta a_2-\alpha a_1)},\qquad a_{21}=\frac{\alpha a_2-\beta a_1}{a_2(\beta a_2-\alpha a_1)}. \tag{20}$$

NSF for x

Thanks to

$$x_1 = x - 2\ln|a_1|, \quad x_{12} = x_1 - 2\ln|a_{12}|,$$

and

$$x_2 = x - 2\ln|a_2|, \quad x_{21} = x_2 - 2\ln|a_{21}|,$$

we obtain

$$x_{12} = x - 2\ln|a_1 a_{12}|, \quad x_{21} = x - 2\ln|a_2 a_{21}|.$$

Therefore

$$x_{12} = x_{21} = 2\ln\left|\frac{\alpha a_2 - \beta a_1}{\beta a_2 - \alpha a_1}\right|.$$

Eliminating a_k $(k = 1, 2)$, we find

$$\boxed{x_{12} = x - 2\ln\left|\frac{\alpha e^{-\frac{1}{2}x_2} - \beta e^{-\frac{1}{2}x_1}}{\beta e^{-\frac{1}{2}x_2} - \alpha e^{-\frac{1}{2}x_1}}\right|.}$$

NSF for u

For the field variable u, we compose the Bäcklund transformations in *two ways* and obtain

$$\boxed{\begin{aligned} u_{12} = {} & \frac{1}{2(\beta a_1 - \alpha a_2)^2(\alpha a_1 - \beta a_2)^2} \\ & \times \Big[(\alpha^2 - \beta^2)(a_1^2 - a_2^2)\big((\alpha a_1 - \beta a_2)^2 + (\beta a_1 - \alpha a_2)^2\big) u_x \\ & \quad + \big((\alpha a_1 - \beta a_2)^4 + (\beta a_1 - \alpha a_2)^4\big) u \\ & \quad + \frac{2(\alpha^2 - \beta^2)}{\alpha\beta}\Big(a_1 a_2 (\alpha a_1 - \beta a_2)^3 - (\beta a_1 - \alpha a_2)^3\Big)\Big] \\ = {} & u_{21}. \end{aligned}}$$

Application

Seed solution: $u = u_0 \neq 0$.
Assume $0 < \alpha^2 u_0^2 < 1$. Define U_1 by $1 - \alpha^2 u_0^2 = U_1^2$.
Real solutions of (14) and (15) may be found either as

$$a_1 = \frac{1 + U_1 \tanh\left(\frac{1}{2} U_1 z_1\right)}{\sqrt{1 - U_1^2}}, \tag{18}$$

or as

$$a_1 = \frac{1 + U_1 \coth\left(\frac{1}{2} U_1 z_1\right)}{\sqrt{1 - U_1^2}}, \tag{19}$$

where $z_1 = x - x_{01} - \dfrac{3 - U_1^2}{\alpha^2} t$.

By means of the BT, we obtain

$$u_1 = \frac{\sqrt{1-U_1^2}}{2\alpha}\left(\frac{\left(1+U_1\tanh\frac{1}{2}U_1z_1\right)^2}{1-U_1^2}+\frac{1-U_1^2}{\left(1+U_1\tanh\frac{1}{2}U_1z_1\right)^2}\right) - \frac{1}{\alpha}\left(\frac{1+U_1\tanh\frac{1}{2}U_1z_1}{\sqrt{1-U_1^2}}+\frac{\sqrt{1-U_1^2}}{1+U_1\tanh\frac{1}{2}U_1z_1}\right), \tag{20}$$

$$x_1 = z_1 + x_{01} + \frac{3-U_1^2}{\alpha^2}t - 2\ln\left|\frac{1+U_1\tanh\frac{1}{2}U_1z_1}{\sqrt{1-U_1^2}}\right|, \tag{21}$$

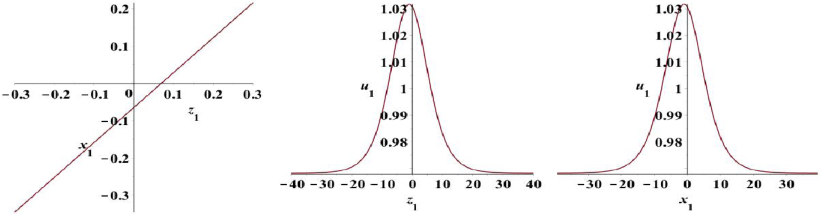
which is a solution of the mCH equation in parametric form.

The solutions may be classified as:

- Soliton

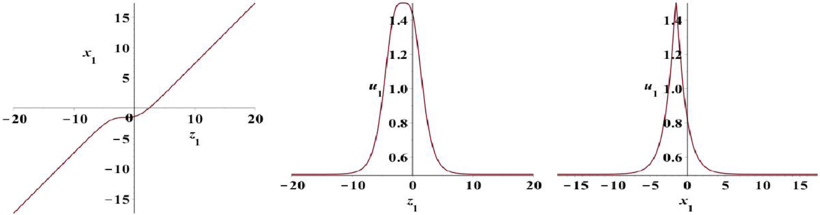
  For $0 < U_1 < \frac{\sqrt{3}}{2}$, the maps from x_1 to z_1 are bijections and equations (20) and (21) supply soliton solutions with speed $c = \frac{3 - U_1^2}{\alpha^2}$.

- Type I singular solution. $U_1 = \frac{\sqrt{3}}{2}$.
- Type II singular solution. $\frac{\sqrt{3}}{2} < U_1 < 1$



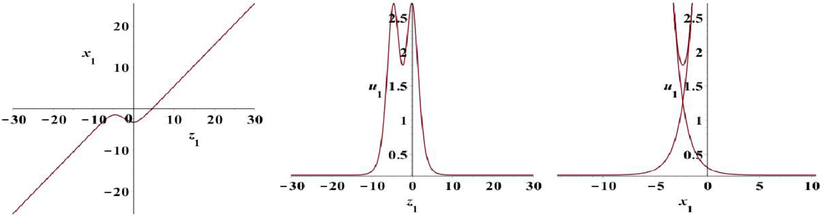
(a) t=0 (b) t=0 (c) t=0

Figure 1. (a)–(c) are respectively the profiles of x_1 as a function of z_1, u_1 as a function of z_1 and one-soliton solution of mCH equation with parameters $t = 0, \alpha = -1, U_1 = \frac{1}{4}, x_{01} = 0, u_0 = \frac{\sqrt{15}}{4}$.



(a) t=0 (b) t=0 (c) t=0

Figure 2. (a)–(c) are respectively the profiles of x_1 as a function of z_1, u_1 as a function of z_1 and the singular solution of mCH equation with parameters $t = 0, \alpha = -1, U_1 = \frac{\sqrt{3}}{2}, x_{01} = 0, u_0 = \frac{1}{2}$.



(a) t=0 (b) t=0 (c) t=0

Figure 3. (a)–(c) are respectively the profiles of x_1 as a function of z_1, u_1 as a function of z_1 and singular soliton solution of mCH equation with parameters $t = 0, \alpha = -1, U_1 = 0.98, x_{01} = 0, u_0 = \frac{3\sqrt{11}}{50}$.

Problms:

- N-BT
  Niu, L, Li (2025)
- Peakons
- Integrable discritizations
- ...

Thank You