Affine Laumon functions and Q-system quivers

Discrete time evolution operators associated with root systems from quiver mutations

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Trivial Example: Plane wave

$$\boxed{e(x|s) := x^{\lambda} = s^{\mu} = q^{\lambda\mu} = e(s|x)}$$

$s = q^{\lambda}$, $x = q^{\mu}$; $q = e^{\hbar}$

Satisfies eigenvalue equation

$$H\, e(x|s) = T_x\, e(x|s) = q^{\lambda} e(x|s) = s\, e(x|s)$$

$\mathbb{T}_1 = \langle x, T_x \rangle / T_x x = q x T_x$ "quantum torus"

Also an eigenfunction of $\gamma(T_x)$:

$$\gamma(T_x)\, x^{\lambda} = \gamma(s)\, x^{\lambda}$$

"Gaussian":

$$\gamma(T_x) = e^{\frac{(\log T_x)^2}{2\hbar}}$$

acts on x, T_x:

$$\gamma(T_x)\, x\, \gamma(T_x)^{-1} = q^{\frac{1}{2}} x T_x$$

commutes with T_x

$$\gamma(s) = q^{\lambda^2/2}$$

$$\boxed{x^{\lambda} : \text{"}q\text{-Whittaker function for } \mathfrak{gl}_1\text{"}}$$

Less trivial Example: sl_2

simple root $\alpha = e_1 - e_2$

Find $\Pi(x_1,x_2|s_1,s_2) = x_1^{\lambda_1} x_2^{\lambda_2} f\left(\frac{x_2}{x_1} \middle| \frac{s_2}{s_1}\right) = x^{\lambda} f(x^{-\alpha}|s^{-\alpha})$

$\quad = 1 + O(x^{-\alpha})$

eigenfunction of $\boxed{g = \gamma(T_1,T_2) \dfrac{1}{\prod_{n\geq 0}(1-q^n \frac{x_2}{x_1})}}$

$\gamma(T_1,T_2) = \prod_{i=1}^{2} \gamma(T_i)$ where $\gamma(T_i) \cdot x_i^m \gamma(T_i)^{-1} = q^{m^2/2} x_i^m$

$\mathbb{T}_2 = \langle x_1, x_2, T_1, T_2 \rangle / T_i x_j = q^{\delta_{ij}} x_j T_i$

$$\boxed{g\,\Pi(x|s) = \gamma(s)\,\Pi(x|s)}$$

- Claim: g Commutes with the difference open Toda Hamiltonian

$$\boxed{H = T_{x_1} + (1 - x_2/x_1)\,T_{x_2}}$$

- Eigenfunctions $\Pi(x|s)$: unique up to normalization
- $H\,\Pi(x|s) = (s_1+s_2)\,\Pi(x|s) \Rightarrow$ "q-Whittaker function"

About q-Whittaker functions

Choose: $\Pi(x|s) = x^{\lambda}(1+\mathcal{O}(x^{-\alpha_i}))$ solution to $H\Pi = e_1(s)\Pi$

"Dual" function: $K(s|x) = \varphi(x^{-\alpha})^{-1}\varphi(s^{-\alpha})^{-1}\,\Pi(x|s)$: $\quad \varphi(x) = \prod_{n\geq 1}(1-q^n x)^{-1}$

$K(s|x) = x^{\lambda}(1+\mathcal{O}(s^{-\alpha}))$

Eigenfunction of

$$D_1(s) = \frac{s_1}{s_1-s_2}T_{s_1} + \frac{s_2}{s_2-s_1}T_{s_2}$$ (q-Whittaker limit of Macdonald op)

$$\boxed{D_1(s)\,K(s|x) = x_1 K(s|x)}$$

The series in s_2/s_1 truncates when $\frac{x_1}{x_2} = q^n$, $n\in\mathbb{Z}_+$

$$K(s|x)\Big|_{x_1=q^n x_2} = K_n(s)$$ is a symmetric polynomial in s_1, s_2.

Constructions of eigenfunctions:

"Givental" or "Mellin-Barnes": [c.f. GKLO, Shapiro-Schrader, ...]

e.g:

$$X^{-\lambda}\,\Pi(X|S) = \varphi\left(x\frac{1}{1-s}T_s\right)\cdot 1 = \sum_{n\geq 0}\frac{\left(x\frac{1}{1-s}T_s\right)^n}{(q)_n}\cdot 1$$

$$= \sum_{n\geq 0}\frac{(x)^n}{(q)_n(s)_n}$$

$x \equiv \frac{x_2}{x_1}$

using non-compact quantum dilogarithm $\varphi(x) = \prod_{n\geq 0}(1-q^n x)^{-1}$

or: $$X^{-\lambda}K(s|x) = \frac{1}{\varphi(s(1-x)T_x)}\cdot 1 = \sum_{n\geq 0}(-1)^n q^{\binom{n}{2}} s^n \frac{(x)_n}{(q)_n}$$

truncates to Class I q-Whittaker

Fermionic construction of eigenfunctions

Using $\gamma(T)\, x^{\lambda} f(\underline{x}|\underline{s}) = \gamma(s)\, x^{\lambda} f(\underline{x},\underline{s})$ $\quad (f = 1 + \Theta(x^{-\alpha}))$

$$\Leftrightarrow \mathcal{J} f(\underline{x}|\underline{s}) = f(\underline{x}|s) \quad \Rightarrow \quad \boxed{\mathcal{J} \overset{\text{def}}{=} x^{-\lambda} \gamma(s)^{-1}\, g\, x^{\lambda}}$$

solution unique up to normalization,

$$\mathcal{J}(\mathcal{J}^{\infty} \cdot 1) = \mathcal{J}^{\infty} \cdot 1 \quad \Rightarrow \quad \mathcal{J}^{\infty} \cdot 1 \text{ is a solution}$$

$\Rightarrow$ "Fermionic formula" for $f(x|s)$

$$\boxed{f(x|s) = \sum_{\{m_i \in \mathbb{Z}_+\}_{i \in \mathbb{N}}}' \frac{q^{\sum_{i} m_i m_j \min(i,j)}}{\prod_i (q)_{m_i}}\, x^{\sum_i m_i}\, s^{\sum i m_i}} \quad *$$

c.f. Hausel [2006]

c.f. $x=1$, Feigin-Stoyanovskii formula for character of principal subspace of $\hat{sl}_2$-mod level : $k = \infty$

c.f. [FFJMM]

Hint about Fermionic formula $\mathcal{T}^{\infty}\cdot 1$:

*

$$\mathcal{T} = \gamma(s|x)\,\varphi(x_2/x_1)$$

$$\gamma(s|T) = \mathrm{Ad}_{x\to\gamma(s)^{-1}}\gamma(T)$$

$$\mathrm{Ad}_{\gamma(s|x)}\, x_i^m = q^{\frac{m^2}{2}}\, s_i^m\, x_i^m\, T_i^m$$

$$\varphi(x) = \prod_{n\geq 0}(1-q^n x)^{-1} = \sum_{n\geq 0}{}' \frac{x^n}{(q)_n}$$

$$\mathcal{T}\cdot 1 = \gamma(s|x)\sum_{m\geq 0}{}' \frac{(x_2/x_1)^m}{(q)_m}\cdot 1$$

$$= \sum_{m\geq 0}{}' \frac{q^{m^2}}{(q)_m}\left(\frac{x_2}{x_1}\right)^m\left(\frac{s_2}{s_1}\right)^m \equiv f^{1}(x|s)$$

$$\mathcal{T}^2\cdot 1 = \sum_{m_1,m_2\geq 0}{}' \frac{q^{m_2^2}}{(q)_{m_1}(q)_{m_2}}\left(\frac{s_2}{s_1}\right)^{m_2}\gamma(s,T_x)\,x_2^{m_1+m_2}\,x_1^{-m_1-m_2}$$

$$= \sum_{m_1,m_2}{}' \frac{(x_2/x_1)^{m_1+m_2}}{(q)_{m_1}(q)_{m_2}}\, q^{m_2^2+(m_1+m_2)^2}\left(\frac{s_2}{s_1}\right)^{m_2+m_1+m_2}\left(\frac{x_2}{x_1}\right)^{m_1+m_2}$$

(level-2)

$$= \sum_{m_1,m_2}{}' \frac{x^{-\alpha_1(m_1+m_2)}}{(q)_{m_1}(q)_{m_2}}\, q^{\sum_{i,j=1}^{2} m_i m_j \min(i,j)}\, s^{-\alpha_1(m_1+2m_2)} \equiv f^{(2)}(x|s)$$

etc...

Where do the time evolution operators g come from?

$\mathfrak{sl}_2$ "Q-system quiver": Q_n $\;①\Longrightarrow②\;$ Q_{n+1} $\qquad \{Q_n\}_{n\in\mathbb{Z}}$ = cluster $\mathcal{A}$-variables

mutations: $\{Q_0,Q_1\} \rightsquigarrow$ sequence $\{Q_n\}_{n\in\mathbb{Z}}$ $\quad n =$ "discrete time"

Time translation: $(Q_n, Q_{n+1}) \longmapsto (Q_{n+1}, Q_{n+2})$

Quiver automorphism: Mutate at vertex 1, permute $1 \longleftrightarrow 2$

(generalized cluster transformation)

$$
\begin{array}{c}
\overset{1}{\circ} \Longrightarrow \overset{2}{\circ} \\
\Big\downarrow \mu_1 \\
\overset{1}{\circ} \Longleftarrow \overset{2}{\circ} \\
\Big\downarrow (12) \\
\overset{2}{\circ} \Longrightarrow \overset{1}{\circ}
\end{array}
$$

$$(Q_n, Q_{n+1}) \xrightarrow{\mu_1} (Q_{n+2}, Q_{n+1}) \xrightarrow{(12)} (Q_{n+1}, Q_{n+2}) = \mathrm{Ad}_g (Q_n, Q_{n+1})$$

defn of g

Review of quantum cluster mutations (simplified)

X_i = Fock Goncharov variables

$$\underset{X_i}{\circ} \xrightarrow{B_{ij}} \underset{X_j}{\circ} \quad \text{means} \quad X_i X_j = q^{-B_{ij}} X_j X_i$$

$$\mu_{X_i}(X_j) = \begin{cases} X_j^{-1}, & i=j \\ \mathrm{Ad}\,\varphi_q(X_i) \circ m_i(X_j) & i \neq j \end{cases}$$

q-dilogarithm $\varphi_q(X_i) = \dfrac{1}{\prod_{n\geq 0}(1-q^n X_i)}$

monomial transformation $m_i(X_j) = q^{\#} X_j X_i^{[B_{ij}]_+}$

$$\mathrm{Ad}\,\varphi(X_i)\,X_j = X_j \begin{cases} \prod_{r=1}^{B_{ij}} (1-q^{-r}X_i)^{-1}, & B_{ij} > 0 \\ \prod_{r=0}^{-B_{ij}-1} (1-q^{r}X_i), & B_{ij} < 0. \end{cases}$$

$\underline{sl_2 \text{ Quiver}}$: $X_1 \; ① \Rightarrow ② \; X_2$ $\qquad X_1X_2 = q^{-2}X_2X_1 \qquad$ (X_i = cluster $\mathcal{X}$-variables)

Time translation operator $g = \sigma_{12}\circ\mu_1 = \sigma_{12}\circ \varphi_q(X_1)\, m_1$

<u>quantum torus</u> $T_i x_j = q^{\delta_{ij}} x_j T_i$, $i,j = 1,2$

Choose X_1, X_2 in quantum torus $\mathbb{T}_2$:

$$\boxed{X_1 =: x^{-\alpha} T^{-\alpha}: \qquad X_2 = x^{\alpha}}$$

$$X_1X_2 = q^{-(\alpha,\alpha)} X_2X_1 = q^{-2}X_2X_1$$

Total monomial transformation on X_i:

$$:(xT)^{-\alpha_1}: = X_1 \xmapsto{m_1} X_1^{-1} \xmapsto{\sigma} :X_2X_1^2: =: (xT^2)^{-\alpha_1}:$$

$$x^{\alpha_1} = X_2 \xmapsto{m_1} :X_2X_1^2: \xmapsto{\sigma} X_1^{-1} =: (xT)^{\alpha_1}:$$

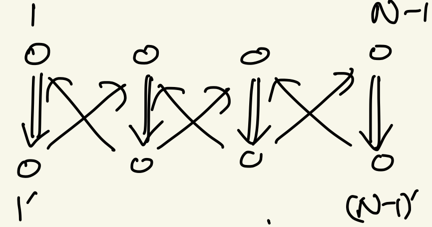
$\Rightarrow m_1 : \quad X_i \mapsto :X_iT_i: \;\Rightarrow\; m_1 = \mathrm{Ad}\,\gamma(T)$

$\qquad \mathrm{Ad}_{\gamma(T)} X_i = q^{1/2} X_i T_i$

$$\Rightarrow \boxed{g = m_1\circ\varphi_q(m_1^{-1}X_1) = \gamma(T)\,\varphi_q(x^{-\alpha_1}) \quad \text{and} \quad g\,\Pi = \gamma(S)\,\Pi}$$

[Di Francesco-K '21]:

Example: $\mathfrak{sl}_N$ Q-system quiver



Time translation [K07]

- mutate at $1, \dots, N-1$
- permute $a \leftrightarrow a'$

Quiver automorphism

Polarization: $X_i = :(xT)^{-\alpha_i}:$, $X_{i'} = x^{\alpha_i}$, $\alpha_i = e_i - e_{i+1},\ i = 1, \dots, N-1$

monomial transformation $X_i \mapsto :X_i T_i:$

$$\Rightarrow \boxed{g = \prod_{i=1}^{N} \gamma(T_i) \prod_{a=1}^{N-1} \varphi(x^{-\alpha_a})}$$

Theorem: [DFK21] Commutes with open Toda $H^{Toda}_{\mathfrak{sl}_N} = T_{x_1} + \sum_{i=1}^{N-1} (1 - x^{-\alpha_i}) T_{i+1}$

and $g\,\Pi(x|s) = \gamma(s)\,\Pi(x|s)$

$$\underline{J = x^{-\lambda}\gamma(s)^{-1} g\, x^{\lambda}}$$

The construction $J^k \cdot 1 = f^{(k)}(x|s)$ as above gives fermionic formula

$$f^{(k)}(x|s) = \sum_{\substack{\{m_i^a \geq 0\}_{a=1,\dots,N-1} \\ i=1,\dots,k}} \prod_{a,i} \frac{x^{-\alpha_a m_i^a} \, s^{-\alpha_a i m_i^a}}{(q)_{m_i^a}} \, q^{\frac{1}{2}\sum_{a,b=1}^{N-1}\sum_{i,j=1}^{k} C_{ab} \min(i,j)\, m_i^a m_j^b}$$

Cartan matrix of sl_N

- $x^{\lambda} f^{(\infty)}(x|s)$ is the eigenfunction of $H^{\mathrm{Toda}}_{sl_N}$, normalized.

- finite k: c.f. Feigen-Stoyanovsky '94 characters of Principal subspaces of level-k modules of sl_N. [Ardonne, K., Stone 2005] (with $x_i = 1$)

- $x \neq 1$: Hausel, 2006: generating functions of Betti numbers of Nakajima Q.V.

Example: $\mathfrak{g} = D, E$

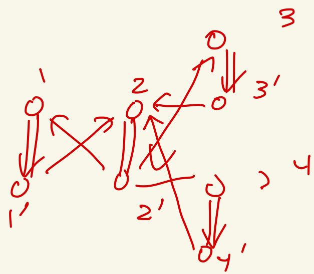
Fock Goncharov variables $\{X_i, X_{i'}\}_{i=1}^{\operatorname{rank}\mathfrak{g}}$

Exchange matrix: $B = \left[\begin{array}{c|c} 0 & C \\ \hline -C & 0 \end{array}\right]_{2N \times 2N}$

Polarization: $X_i = (xT)^{-\alpha_i}$, $X_{i'} = x^{\alpha_i}$ $\Rightarrow$ $X_i X_{j'} = q^{-(\alpha_i,\alpha_j)} X_{j'} X_i$

$X_i X_j = X_j X_i$ and $X_{i'} X_{j'} = X_{j'} X_{i'}$

e.g. D_4:



Time translation: • mutate at X_i
• Permute $i \leftrightarrow i'$

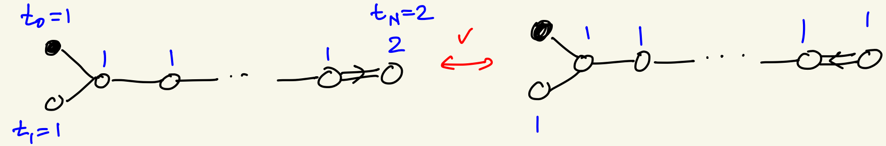
Quiver automorphism

$$\Rightarrow \quad g = \gamma(T_1, \dots, T_N) \prod_{i=1}^{r} \varphi(x^{-\alpha_i}) \qquad [K07]$$

commutes with difference Toda Hamiltonians

Other root systems Affine Dynkin diagram and their duals:

Ex:

$B_N^{(1)}$  $A_{2N-1}^{(2)}$

$C_N^{(1)}$  $D_{N+1}^{(2)}$

$G_2^{(1)}$  $D_4^{(3)}$

etc...

- Integers t_a, $t_a^\vee$ associated with $\mathfrak{g}$, $\mathfrak{g}^\vee$ ($t_a = 1$ for most twisted types)
- $\alpha_i^\vee = t_i \alpha_i$
- Erase zero node
- for Q-system quiver

$X_N^{(r)}$	$(X_N^{(r)})^\vee$	R	t_a	$t_a^\vee$	$C^T = (T^\vee)^{-1}C'T$
$B_N^{(1)}$	$A_{2N-1}^{(2)}$	B_N	$(1, ..., 1, 2)$	$(1, ..., 1)$	C_N
$C_N^{(1)}$	$D_{N+1}^{(2)}$	C_N	$(2, ..., 2, 1)$	$(1, ..., 1)$	B_N
$F_4^{(1)}$	$E_6^{(2)}$	F_4	$(1, 1, 2, 2)$	$(1, 1, 1, 1)$	F_4^T
$G_2^{(1)}$	$D_4^{(3)}$	G_2	$(1, 3)$	$(1, 1)$	G_2^T
$A_{2N-1}^{(2)}$	$B_N^{(1)}$	C_N	$(1, ..., 1)$	$(1, ..., 1, 2)$	C_N
$D_{N+1}^{(2)}$	$C_N^{(1)}$	B_N	$(1, ..., 1)$	$(2, ..., 2, 1)$	B_N
$E_6^{(2)}$	$F_4^{(1)}$	$F_4^\vee$	$(1, 1, 1, 1)$	$(1, 1, 2, 2)$	F_4^T
$D_4^{(3)}$	$G_2^{(1)}$	$G_2^\vee$	$(1, 1)$	$(1, 3)$	G_2^T

Q-system quiver from $\{\alpha_i, \alpha_i^\vee\}_0$: $X_i X_j = q^{-B'_{ij}} X_j X_i = q_i^{-B_{ij}} X_j X_i$

"symmetrized" $B'_{ij} = d_i B_{ij}$ $\qquad q_i = q^{d_i}$

- $C'_{ab} = (\alpha_a, \alpha_b)$ $\qquad \big((\alpha_a, \alpha_a) = 2$ for long roots$\big)$ $\qquad a, b = 1, \dots, N$

  Symmetric matrix $\qquad$ (drop α_0)

- Skew symmetric matrix $B' = \begin{bmatrix} [T, C'] & C'T \\ -TC' & 0 \end{bmatrix} \overset{\text{Def}}{=} \begin{bmatrix} T^\vee & 0 \\ 0 & T^\vee \end{bmatrix} B$

  $T = \mathrm{diag}(t_a)$, $T_a^\vee = \mathrm{diag}(t_a^\vee)$

$$\Rightarrow B = \begin{bmatrix} C - C^T & C^T \\ -C & 0 \end{bmatrix} \qquad C = (T^\vee)^{-1} C' T = \begin{cases} C_{\mathfrak{g}_0}^T & \text{in untwisted types} \\ C_{\mathfrak{g}_0} & \text{in twisted types} \end{cases}$$

B is the exchange matrix for the cluster algebra.

Example: $B_2^{(1)}$:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 2 \arrow[d, shift left] \arrow[d, shift right] \\ 1' \arrow[ur] & 2' \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

Example: $A_3^{(2)}$

$$B = \left(\begin{array}{c|c} 0 & C_{C_2} \\ \hline -C_{C_2} & 0 \end{array}\right)_{4\times 4}$$

$C_{C_2} = C_2$ Cartan matrix

(valued quiver)

$$B = \left[\begin{array}{cc|cc} 0 & & 2 & -2 \\ & & -1 & 2 \\ \hline -2 & 2 & 0 & \\ 1 & -2 & & \end{array}\right] \qquad D = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 1 & \\ & & & 2 \end{bmatrix}$$

$t_a^\vee$ for type B_2

Root Data $(\alpha_i, t_i, t_i^\vee)_{i=1,\dots,N}$ $\rightsquigarrow$ Cluster variables in $\mathbb{T}_N$

$$X_a = x^{-t_a\alpha_a} T^{-\alpha_a}, \quad X_{a'} = x^{t_a\alpha_a} \quad \Rightarrow \quad X_i X_j = q_i^{-B_{ij}} X_j X_i \qquad q_i = q^{t_i^\vee}$$

$\rightsquigarrow$ Defines cluster algebra.

Integrable evolution: 1 step time translation:

If $t_a = 1$ for all $a = 1, \dots, N$: $(A_N^{(1)}, D_N^{(1)}, E_i^{(1)}, X_N^{(r)}$ with $r > 1)$

1. mutate all vertices $a = 1, \dots, N$
2. permute $a \leftrightarrow a'$ Quiver automorphism.

Time translation operator: $g = \gamma_q(T_1, \dots, T_N) \prod_{a=1}^{N} \varphi_{q^{t_a^\vee}}(x^{-\alpha_a})$

([DFK21] in classical types)

Non-simply laced types: $t = \max_a \{t_a\} > 1$

① Repeat t times:
- mutate at a for a short
- permute $a \leftrightarrow a'$

② mutate at $a =$ long roots, permute $a \leftrightarrow a'$

$\Rightarrow$ Time translation operator:

$$g = \left(\gamma(T_1, \dots, T_N)^{1/t} \prod_{a \text{ short}} \varphi_q(X^{-t_a \alpha_a}) \right)^t \prod_{a \text{ long}} \varphi_q(X^{-\alpha_a})$$

Thm:
([DFK21] for classical types)

- g commutes with difference "Toda Hamiltonians"
- Common eigenfunctions = "q-Whittaker functions".
- Eigenvalue of $g(x) = \gamma(s)$

Remark: The construction $f(X|S) = \mathcal{T}^{\infty} \cdot 1$ gives the fermionic form of the Feigin-Stoyanovski type "Gordon filtration"

[e.g Ardonne-K.-Stone, see also Hatayama et al]

Quadratic form : $\frac{1}{2} \sum_{\substack{a,b=1 \\ i,j \geq 1}}^{N} {}' \, m_i^{(a)} m_j^{(b)} \min(t_b i, t_a j) C_{ab}$

What about affine root systems? (e.g. closed Toda) $A^{(1)}_{N-1}$:

Extended root system

$$\hat{\alpha}_i = e_i - e_{i+1} \quad (i=1,\dots,N-1), \quad \hat{\alpha}_N = \delta + e_N - e_1, \; d \quad (d,\delta)=1$$

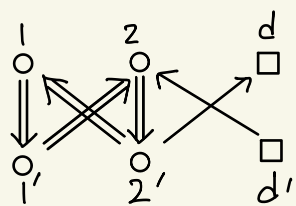
Extended Cartan matrix $\tilde{C}_{N+1}$

Extended q-Torus

$$\hat{T}_N = \langle x_1, \dots, x_N, x^{-\delta}\equiv p \,;\, T_1,\dots,T_N, T_{p,\kappa}\rangle \Big/ \; T_i x_j = q^{\delta_{ij}} x_j T_i \,, \quad T_{\kappa,p}\, p = \kappa p\, T_{\kappa p}$$

Cluster variables: $X_a = :(xT)^{-\alpha_a}:$, $\quad X_{a'} = x^{\alpha_a} \quad (a \in \mathbb{Z}/N\mathbb{Z})$ + coefficients

Example $\hat{sl}_2$:



$$\rightsquigarrow \; B = \begin{pmatrix} 0 & \tilde{C} \\ -\tilde{C} & 0 \end{pmatrix} \; : \; \tilde{C} = \begin{array}{ccc} 1 & 2 & d \\ \end{array} \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

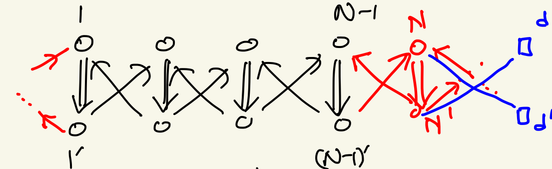
Time evolution:

$$\pi(a,a') \circ \mu_2 \circ \mu_1 \quad \text{quiver automorphism}$$

acts on $X_a, X_{a'}$ as the adjoint action of $\hat{g} = \hat{\gamma}(T) \cdot \varphi(x_2/x_1)\varphi(p\, x_1/x_2)$

$$\hat{\gamma}(T) = T_{p,\kappa}\gamma(T_1)\gamma(T_2) : \quad x^{-\alpha_1} \mapsto :(xT)^{-\alpha_1}: , \quad x^{-\alpha_0} = p\, x^{\alpha_1} \mapsto \kappa p (xT)^{\alpha_1}$$

For $\widehat{sl}_N$, $\quad \hat{g} = \hat{\gamma}(T)\prod_{i=1}^{N}\varphi(x^{-\alpha_i})$ $\quad$ with $x^{-\alpha_N} = p\,\frac{x_1}{x_N}$



"Closed Toda Hamiltonian"

$$\hat{H}(p) = \sum_{i=0}^{N-1}(1-x^{-\alpha_i})T_{i+1} = \left(1-p\frac{x_1}{x_N}\right)T_1 + \sum_{i=1}^{N-1}\left(1-\frac{x_{i+1}}{x_i}\right)T_{i+1}$$

$\kappa \neq 1$: Does not commute with $\hat{g}$: $\hat{g}\,\hat{H}(p) = \hat{H}(\kappa p)\,\hat{g}$

The eigenfunctions of $\hat{g}$ are "affine Laumon functions"

$$\hat{g}\, x^{\lambda} f(x,p\,|\,s,\kappa) = \gamma(s)\, x^{\lambda} f(x,p\,|\,s,\kappa), \qquad f(x,p\,|\,s,\kappa) = 1 + O(x^{-\alpha_i})$$

$$\hat{J} = x^{-\lambda}\gamma(s)^{-1}\,\hat{g}\,x^{\lambda} \quad : \quad \hat{J} f(x,p\,|\,s,\kappa) = f(x,p\,|\,s,\kappa).$$

The affine Laumon function $f(x, p | s, \kappa)$

Explict formula: Shiraishi [2019] using Nekrasov function

$\sim$ Givental type formula (closed)

Fermionic type formula: $\hat{J}^{\infty} \cdot 1$ (converges $\kappa \neq 1$):

has quadratic form $\frac{1}{2} \sum m_i^{(a)} m_j^{(b)} C_{ab} \min(i,j)$

$C =$ affine Cartan matrix (singular)

- c.f. Hausel's function: generating f^n for Betti numbers for cyclic Nakajima quiver variety.

Summary:

① For any root data $\{\alpha_i; t_i, t_i^\vee\}_{i\in I}$ $\rightsquigarrow$ Q-system quantum cluster alg

Evolution operator: Generalized mutation sequence

② Choice of polarization for cluster variables $\rightsquigarrow$ $\mathcal{G}$: q-difference operator, eigenfunctions $x^\lambda(1+O(x^{-\alpha_i}))$

③ Eigenfunctions are q-Whittaker function in "open" case

④ Fermionic expressions $x^\lambda \mathcal{T}^\infty \cdot 1$

Thank you organizers!