q-character theory for quantum symmetric pairs

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Quantum loop algebras

Quantum loop algebras

- $\mathbf{k} = \mathbb{C}$, and q is not a root of unity
- $\mathfrak{g}$ - simple Lie algebra (e.g., $\mathfrak{sl}_{n+1}(\mathbb{C})$)
- $\widehat{\mathfrak{g}}$ - associated affine Kac–Moody algebra

$$0 \to \mathbb{C}\mathbf{c} \to \widehat{\mathfrak{g}} \to \mathfrak{g}[t^{\pm 1}] \to 0$$

- $U_q(\widehat{\mathfrak{g}})$ - quantum affine algebra
- $U_q(L\mathfrak{g})$ - quantum loop algebra = quantum affine algebra of level 0 - it has finite-dimensional representations

3 presentations

- FRT:

$$RTT = TTR,$$

- Drinfeld–Jimbo: $E_i, F_i, K_i^{\pm 1}$ + quantum Serre relations,
- Drinfeld's new presentation.

Drinfeld vs Drinfeld–Jimbo

Drinfeld generators

DJ generators

$$
\begin{array}{cccc}
 & \vdots & & \vdots \\
 & x^{\pm}_{1,1}\ \ \phi^{+}_{1,1} & \cdots & x^{\pm}_{n,1}\ \ \phi^{+}_{n,1} \\
x^{\pm}_{0,0}\ \ \phi^{\pm}_{0,0} & x^{\pm}_{1,0}\ \ \phi^{\pm}_{1,0} & \cdots & x^{\pm}_{n,0}\ \ \phi^{\pm}_{n,0} \\
 & x^{\pm}_{1,-1}\ \ \phi^{-}_{1,-1} & \cdots & x^{\pm}_{n,-1}\ \ \phi^{-}_{n,-1} \\
 & \vdots & & \vdots
\end{array}
$$

- $x^+_{i,0} = E_i$, $x^-_{i,0} = F_i$ and $\phi^{\pm}_{i,0} = K_i^{\pm 1}$
- Two **triangular decompositions**:

$$
\begin{aligned}
\text{DJ}: &\quad \langle F_i \rangle \otimes \langle K_i^{\pm 1} \rangle \otimes \langle E_i \rangle, \\
\text{Drinfeld}: &\quad \langle x^-_{i,r} \rangle \otimes \langle \phi^{\pm}_{i,r} \rangle \otimes \langle x^+_{i,r} \rangle.
\end{aligned}
$$

Braid group action

Lusztig's braid group action

For each $i \in \mathbb{I}$, there is an automorphism T_i of $U_q(L\mathfrak{g})$. These satisfy braid group relations.

Example: $U_q(L\mathfrak{sl}_n)$

Action on Drinfeld–Jimbo generators:

$$T_i(E_j) = \begin{cases} [E_i, E_j]_{q^{-1}} & a_{ij} = -1, \\ -F_i K_i & a_{ij} = 2, \\ E_j & a_{ij} = 0, \end{cases} \qquad T_i(F_j) = \begin{cases} [F_j, F_i]_q & a_{ij} = -1, \\ -K_i^{-1} E_i & a_{ij} = 2, \\ F_j & a_{ij} = 0. \end{cases}$$

Reduced expressions for fundamental weights:

$$\omega_i = \pi^i (s_{n-i} \cdots s_{n-1}) \cdots (s_2 \cdots s_{i+1})(s_1 \cdots s_i).$$

Drinfeld generators

- Drinfeld generators are obtained from Drinfeld–Jimbo generators via the braid group action:
  - **Real** root vectors:

$$x_{i,r}^{+} = o(i)^r T_{\omega_i}^{-r}(E_i), \qquad x_{i,r}^{-} = o(i)^r T_{\omega_i}^{r}(F_i).$$

  - **Imaginary** root vectors: $\phi_{i,r}^{\pm}$ (for $r \geq 0$) - fixed by T_{ω_j}.
- The subalgebra

$$\langle K_i^{\pm 1} \rangle \subset \langle \phi_{i,r}^{\pm} \rangle \subset U_q(L\mathfrak{g})$$

is commutative. We will call $\langle \phi_{i,r}^{\pm} \rangle$ the **Drinfeld–Cartan subalgebra**.

Example: relations in $U_q(L\mathfrak{sl}_2)$

The algebra $U_q(L\mathfrak{sl}_2)$ is generated by

$$\mathbf{x}^{\pm}(z) = \sum_{r\in\mathbb{Z}} x_r^{\pm} z^r, \qquad \phi^{\pm}(z) = \sum_{r\geq 0} \phi_{\pm r} z^{\pm r},$$

subject to the relations:

$$\begin{aligned}
\phi^{\star}(z)\phi^{*}(w) &= \phi^{*}(w)\phi^{\star}(z), \\
\phi_0^{\star}\mathbf{x}^{\pm}(z) &= q^{\star\pm 2}\mathbf{x}^{\pm}(z)\phi_0^{\star}, \\
\phi^{\star}(z)\mathbf{x}^{\pm}(w) &= \frac{q^{\pm 2}w - z}{w - q^{\pm 2}z}\mathbf{x}^{\pm}(w)\phi^{\star}(z), \\
\mathbf{x}^{\pm}(z)\mathbf{x}^{\pm}(w) &= \frac{q^{\pm 2}w - z}{w - q^{\pm 2}z}\mathbf{x}^{\pm}(w)\mathbf{x}^{\pm}(z), \\
[\mathbf{x}^{+}(z), \mathbf{x}^{-}(w)] &= \frac{\delta(w/z)}{q - q^{-1}}\left(\phi^{+}(w) - \phi^{-}(z)\right).
\end{aligned}$$

Finite-dimensional reps

- Highest **loop** weight/Verma theory

$$\mathbf{x}_i^{+}(z)\cdot v_0 = 0, \qquad \phi_i^{\pm}(z)\cdot v_0 = \boldsymbol{\lambda}_i^{\pm}(z)v_0.$$

- Let V be a simple FD $U_q(\widehat{\mathfrak{g}})$-module. Eigenvalues on a HW vector v_0 can be described in terms of *Drinfeld polynomials*:

$$\boldsymbol{\lambda}_i^{\pm}(z) = \sum_{k=0}^{\infty} \lambda_{i,\pm k}^{\pm} z^{\pm k} = q^{\deg P_i}\frac{P_i(q^{-1}z)}{P_i(qz)}.$$

- There is a bijective correspondence (Chari–Pressley)

$$\{\text{ simple FD modules }\} \longleftrightarrow \{\text{ tuples of polynomials with cnst term } 1\}.$$

- More generally (Frenkel–Reshetikhin), the generalized eigenvalues are expansions of rational functions of the form

$$\gamma_i^{\pm}(z) = q^{\deg Q_i - \deg R_i}\frac{Q_i(q^{-1}z)R_i(qz)}{Q_i(qz)R_i(q^{-1}z)}.$$

q-characters

Classical character theory

- $\mathfrak{g}$ - simple complex Lie algebra
- G - corresponding simply-connected Lie group
- $T \subset G$ - maximal torus, W - Weyl group

$$\{ \text{ FD simple modules } \} \longleftrightarrow \{ \text{ dominant integral weights } \}$$

- For classification, only information about the HW space is needed.
- What about an invariant, which encodes information about every weight space?

$$\begin{aligned} \chi\colon [\mathrm{Rep}^{fd}\, G] \ &\to \ \mathbb{Z}[T] = \mathbb{Z}[y_1^{\pm 1}, \cdots, y_n^{\pm 1}] \\ V &\mapsto \chi_V(t) = \mathrm{Tr}_V(t) \end{aligned}$$

with

$$\operatorname{im} \chi = \mathbb{Z}[T]^W.$$

q-characters

The original construction of q-characters was proposed by Frenkel and Reshetikhin (and Knight - for Yangians).

- The usual character χ encodes the eigenvalues of the $K_i^{\pm}$.
- There exists q-character χ_q, encoding the eigenvalues of the $\phi_i^{\pm}(z)$, which lifts χ to the affine case:

$$\begin{array}{ccc} [\operatorname{Rep} U_q(L\mathfrak{g})] & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I_0, a\in\mathbb{C}^\times} \\ \downarrow{\scriptstyle res} & & \downarrow \\ [\operatorname{Rep} U_q(\mathfrak{g})] & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i\in I_0} \end{array}$$

There are four constructions of q-characters:

- integrable systems: the universal R-matrix
- algebraic: via Drinfeld's new presentation
- geometric: via Nakajima's quiver varieties
- axiomatic

Universal R-matrix

- The quantum group $U_q(L\mathfrak{g})$ is a **quasi-triangular Hopf algebra**, i.e., there exists a **universal R-matrix** $R \in U_q(L\mathfrak{g}) \widehat{\otimes} U_q(L\mathfrak{g})$ such that

$$R\Delta(x) = \Delta^{op}(x)R \qquad (x \in U_q(L\mathfrak{g})),$$
$$(\Delta \otimes \mathrm{id})R = R_{13}R_{23}, \qquad (\mathrm{id} \otimes \Delta)R = R_{13}R_{12}.$$

- Let (V, π_V) be a finite-dimensional representation. The associated **L-operator** and **transfer matrix** are

$$L_V(z) = (\pi_{V(z)} \otimes \mathrm{id})(R), \qquad t_v = \mathrm{Tr}\, q^{2\rho} L_V(z).$$

- KTLSSD **factorization** of the universal R-matrix:

$$R = R^+ R^0 R^- T, \qquad R^0 = \exp\left(- \sum_{n \geq 0} \frac{n(q - q^{-1})}{[n]_q} \sum_{i \in I_0} h_{i,n} \otimes \widetilde{h}_{i,-n} z^n \right).$$

q-characters & universal R-matrix

- The **q-character map** can be defined as the composition

$$\chi_q\colon [\mathrm{Rep}\, U_q(L\mathfrak{g})] \xrightarrow{t_v} U_q(\mathfrak{b}_-)[[z]] \xrightarrow{HC} U_q(\widetilde{\mathfrak{h}})[[z]].$$

- The image is contained in the polynomial ring generated by

$$Y_{i,a} = K_{\omega_i}^{-1} \exp\left(\sum_{k>0} \tilde{h}_{i,-k} a^k z^k\right),$$

with

$$\phi_i^{\pm}(z) = K_i^{\pm 1} \exp\left(\pm(q-q^{-1}) \sum_{k=1}^{\infty} h_{i,\pm k} z^{\pm k}\right),$$

$$\tilde{h}_{i,-k} = \sum_{j \in I_0} \widetilde{C}_{ji}(q^k) h_{j,-k}.$$

q-characters & Drinfeld presentation

- Let V be a FD $U_q(L\mathfrak{g})$-module with a generalized eigenvalue decomposition wrt $\langle \phi^{\pm}_{i,r} \rangle$

$$V = \bigoplus_{\gamma} V_{\gamma}.$$

- There is a correspondence

$$\{ \text{ monomials in } Y_{i,a} \} \longleftrightarrow \{ \text{ eigenvalues } \}.$$

- The eigenvalues are expansions of rational functions of the form

$$\gamma_i^{\pm}(z) = q^{\deg Q_i - \deg R_i} \frac{Q_i(q^{-1}z) R_i(qz)}{Q_i(qz) R_i(q^{-1}z)}.$$

- Writing $Q_i(z) = \prod_{r=1}^{k_i} (1 - z a_{i,r})$, $R_i(z) = \prod_{s=1}^{l_i} (1 - z b_{i,s})$, the q-character of V can now be expressed as

$$\chi_q(V) = \sum_{\gamma} \dim(V_{\gamma}) M_{\gamma}, \qquad M_{\gamma} = \prod_{i \in I_0} \prod_{r=1}^{k_i} Y_{i,a_{i,r}} \prod_{s=1}^{l_i} Y^{-1}_{i,b_{i,s}}.$$

The coproduct and multiplicativity

Multiplicativity

The q-character map is a *ring* homomorphism, i.e.,

$$\chi_q(V \otimes W) = \chi_q(V) \cdot \chi_q(W).$$

This is immediate from the R-matrix construction, but non-trivial from the "Drinfeld presentation" point of view.

Chari-Pressley and Damiani proved that

- $\phi_i^{\pm}(z)$ are *approximately group-like*:

$$\Delta(\phi_i^{\pm}(z)) \equiv \phi_i^{\pm}(z) \otimes \phi_i^{\pm}(z) \qquad \text{mod} \quad U_+ \otimes U_-,$$

- $x_{i,\geq 0}^{+}(z)$, etc., are *approximately twisted-primitive*:

$$\Delta(x_{i,\geq 0}^{+}(z)) \equiv x_{i,\geq 0}^{+}(z) \otimes \phi_i^{\pm}(z) + 1 \otimes x_{i,\geq 0}^{+}(z) \quad \text{mod} \quad U_{\geq 2} \otimes U_-,$$

in the Drinfeld gradation ($\deg x_{i,r}^{+} = \alpha_i$, $\deg x_{i,r}^{-} = -\alpha_i$, $\deg h_{i,r} = 0$).

Applications and further results

- The map $\chi_q \colon [\operatorname{Rep} U_q(L\mathfrak{g})] \to \mathbb{Z}[Y_{i,a}^{\pm 1}]$ is injective.
- Can be computed in many cases via the Frenkel–Mukhin algorithm.
- Nakajima, Hernandez: t-deformation, used to compute multiplicities $[M_P : L_Q]$
- Original motivation: deformed W-algebras.
- Kirillov–Reshetikhin conjecture, relation to Q- and T-systems, Bethe ansatz.
- Connection to cluster algebras.

Quantum symmetric pairs

Quantum symmetric pairs

- Classical symmetric pair: ss. Lie algebra $\mathfrak{g}$ + involution σ, e.g.,
  - $\mathfrak{g} = \mathfrak{sl}_n$, $\sigma(x) = -x^t \quad \rightsquigarrow \quad \mathfrak{g}^\sigma = \mathfrak{so}_n$,
  - $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, $\sigma(x) = -J^{-1}x^tJ \quad \rightsquigarrow \quad \mathfrak{g}^\sigma = \mathfrak{sp}_n$.
- **Origin** in the classical case: real forms of Lie groups, classification of symmetric spaces.
- **Problem**: classically,

  $$U(\mathfrak{g}^\sigma) \subset U(\mathfrak{g}).$$

  But

  $$U_q(\mathfrak{g}^\sigma) \not\subset U(\mathfrak{g}).$$

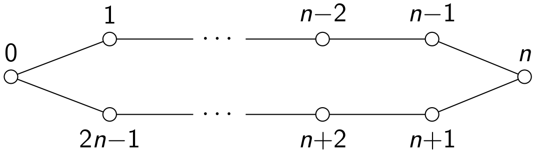
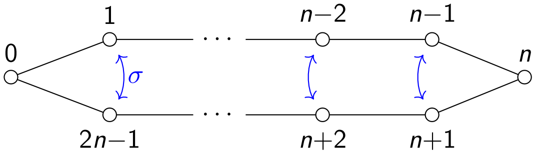
- **Solution**: there exists an alternative *compatible quantization* - coideal subalgebra $\mathbf{U}^\imath = U'_q(\mathfrak{g}^\sigma) \subset U_q(\mathfrak{g})$:

  $$\Delta(\mathbf{U}^\imath) \subset \mathbf{U}^\imath \otimes U_q(\mathfrak{g}), \qquad \mathbf{U}^\imath \xrightarrow{q\to 1} U(\mathfrak{g}^\sigma).$$

- There is a monoidal action: $\operatorname{Rep} \mathbf{U}^\imath \curvearrowleft \operatorname{Rep} U_q(\mathfrak{g})$.

Quantum symmetric pairs - types

Quantum symmetric pairs are classified by **Satake diagrams**, encoding the generators $B_i = F_i - E_{\sigma(i)} K_i^{-1}$.

Type	diagram	fin. class. limit
AI		$\mathfrak{so}_{2n}$
AIII		$\mathfrak{s}(\mathfrak{gl}_n^{\oplus 2})$

QSP - motivation

1. Quantum Schur–Weyl duality

$$\begin{array}{lccccc} \text{type A:} & U_q(\mathfrak{sl}_n) & \curvearrowright & V_q^{\otimes m} & \curvearrowleft & \mathcal{H}(\mathrm{A}_{m-1}) \\ & \cup & & & & \cap \\ \text{type B:} & \mathbf{U}^\imath(\mathrm{AIII}) & \curvearrowright & V_q^{\otimes m} & \curvearrowleft & \mathcal{H}(\mathrm{B}_m). \end{array}$$

2. Gelfand–Zetlin patterns

$$U_q(\mathfrak{sl}_n) \subset U_q(\mathfrak{sl}_{n+1}) \subset U_q(\mathfrak{sl}_{n+2}) \subset \dots$$
$$U_q(\mathfrak{so}_n) \not\subset U_q(\mathfrak{so}_{n+1}) \not\subset U_q(\mathfrak{so}_{n+2}) \not\subset \dots$$
$$U_q'(\mathfrak{so}_n) \subset U_q'(\mathfrak{so}_{n+1}) \subset U_q'(\mathfrak{so}_{n+2}) \subset \dots$$

3. Evaluation homomorphisms

$$U_q(L\mathfrak{sl}_n) \to U_q(\mathfrak{gl}_n)$$
$$Y_q^{tw}(\mathfrak{so}_n) \to U_q'(\mathfrak{so}_n)$$
$$Y_q^{tw}(\mathfrak{sp}_{2n}) \to U_q'(\mathfrak{sp}_{2n})$$

New Drinfeld presentation for affine QSP

- Rank one case (q-Onsager algebra) - Kolb–Baseilhac (2017).
- General split simply-laced case (Satake = Dynkin diagram, AI) - Lu–Wang (2020).
- Non-simply laced, quasi-split cases ($AIII$) - Lu, Pan, Wang, Zhang (2021-25).

New braid group action (Kolb, Pellegrini, Lu, Wang, Zhang)

1. There exists an action $s_i \mapsto \mathbf{T}_i$ of the (relative) braid group on $\mathcal{B}$.
2. Setting $A_{i,r} = \mathbf{T}_{\omega_i}^{-r}(B_i)$ yields Drinfeld-type generators.
3. The Drinfeld-type presentation exhibits an infinitely generated commutative subalgebra

$$\mathbb{H} = \langle H_{i,s} \rangle_{i \in I_0, s > 0} = \langle \Theta_{i,s} \rangle_{i \in I_0, s \geq 0}.$$

Key fact

In general, $\mathbf{T}_i(B_j) \neq T_i(B_j)$. In other words, the braid group action on $\mathcal{B}$ is not the restriction of the usual braid group action on $U_q(\widehat{\mathfrak{g}})$.

Example: q-Onsager algebra

- Originally defined by P. Terwilliger (coming from combinatorics, tridiagonal pairs).
- $\mathcal{O}_q$ is a coideal subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$.
- Generated by

$$B_i = F_i - c_i E_i K_i^{-1} + s_i K_i^{-1} \quad (i = 0, 1).$$

- Relations:

$$B_1^3 B_0 - [3] B_1^2 B_0 B_1 + [3] B_1 B_0 B_1^2 - B_0 B_1^3 = q[2]^2 c_1 (B_0 B_1 - B_1 B_0)$$
$$B_0^3 B_1 - [3] B_0^2 B_1 B_0 + [3] B_0 B_1 B_0^2 - B_1 B_0^3 = q[2]^2 c_0 (B_1 B_0 - B_0 B_1).$$

- The parameters s_i do not change the algebra structure but affect the coideal structure.

q-Onsager algebra in the loop presentation

The q-Onsager algebra is isomorphic to the algebra generated by coefficients of the series

$$\mathbf{A}(z) = \sum_{r\in\mathbb{Z}} A_r z^r, \qquad \mathbf{\Theta}(z) = \sum_{r\geq 0} \Theta_r z^r,$$

subject to the following relations:

$$\begin{aligned}
\mathbf{\Theta}(z)\mathbf{\Theta}(w) &= \mathbf{\Theta}(w)\mathbf{\Theta}(z),\\
\mathbf{\Theta}(z)\mathbf{A}(w) &= \frac{\mu(w,z)}{\mu(z,w)}\frac{\mu(z,w^{-1})}{\mu(w^{-1},z)}\mathbf{A}(w)\mathbf{\Theta}(z),\\
\mathrm{Sym}_{z,w}\,\mu(z,w)\mathbf{A}(z)\mathbf{A}(w) &= (w-z)\delta(zw)(\mathbf{\Theta}(z)-\mathbf{\Theta}(w)),
\end{aligned}$$

where $\mu(z,w) = q^2 z - w$.

q-characters for QSP

Questions

Assumption: Let $(\mathbf{U}^\imath, U_q(\widehat{\mathfrak{g}}))$ be a quasi-split quantum symmetric pair (precisely: ABCD split or AIII).

- What is the relationship between the Drinfeld generators of $\mathbf{U}^\imath$ and $U_q(\widehat{\mathfrak{g}})$?

$$\begin{array}{ccc}
\mathbf{U}^\imath & \xhookrightarrow{B_i \mapsto F_i - E_{\sigma(i)} K_i^{-1}} & U_q(\widehat{\mathfrak{g}}) \\
B_i \rightsquigarrow \mathbf{T}'_{\omega_i}(B_i) \Big\| & & \Big\| x_i^\pm \rightsquigarrow T'_{\omega_i}(x_i^\pm) \\
\mathbf{U}^\imath{}_{Dr} & \xhookrightarrow{?} & U_q(\widehat{\mathfrak{g}})_{Dr}
\end{array}$$

- How do the Drinfeld generators of $\mathbf{U}^\imath$ behave under coproduct?

$$\begin{array}{ccc}
\mathbf{U}^\imath & \xrightarrow{B_i \mapsto 1 \otimes B_i + B_i \otimes K_i^{-1}} & \mathbf{U}^\imath \otimes U_q(\widehat{\mathfrak{g}}) \\
B_i \rightsquigarrow \mathbf{T}'_{\omega_i}(B_i) \Big\| & & \Big\| \\
\mathbf{U}^\imath{}_{Dr} & \xrightarrow{?} & \mathbf{U}^\imath{}_{Dr} \otimes U_q(\widehat{\mathfrak{g}})_{Dr}
\end{array}$$

Main structural results

Factorization theorem (P., Li–P.)

The generating series $\mathbf{\Theta}_i(z)$ have the following *factorization property*:

$$\mathbf{\Theta}_i(z) \equiv \phi_i^-(z^{-1})\phi_i^+(z) \qquad \text{mod } U_q(\widehat{\mathfrak{g}})_+[[z]].$$

Coproduct theorem (P., Li–P.)

The generating series $\mathbf{\Theta}_i(z)$ are *approximately group-like*:

$$\Delta(\mathbf{\Theta}_i(z)) \equiv \mathbf{\Theta}_i(z) \otimes \mathbf{\Theta}_i(z) \qquad \text{mod } \mathbf{U}^\imath \otimes U_q(\widehat{\mathfrak{g}})_+[[z]].$$

Rough idea of proof:

- reduction to rank 1 cases,
- comparison of lattice operators $\mathbf{T}_{\omega_i}$ and T_{ω_i} in the two braid group actions.

Boundary q-characters

How to define q-characters for QSP?

1. **Geometrically?** We could, building on Su–Wang, Luo–Su–Xu, but this is so far restricted to type AIII.
2. **Universal K-matrix?** We could, building on Appel–Vlaar, but we do not know an analogue of the KTLSSD factorization to compute such characters.
3. **Drinfeld presentation?** Yes - the approach we follow.
   - $V = \bigoplus_\gamma V_\gamma$ - FD $\mathbf{U}^\imath$-module with a generalized eigenvalue decomposition wrt $\langle \Theta_{i,r} \rangle$
   - $\mathbf{Y}_{i,a} = Y_{i,a} Y^{-1}_{\sigma(i),a^{-1}}$
   - There is (roughly) a correspondence

$$\{ \text{ monomials in } \mathbf{Y}_{i,a} \} \longleftrightarrow \{ \text{ eigenvalues } \}.$$

   - $\chi_q^b(V) = \sum_\gamma \dim(V_\gamma) M_\gamma$

Boundary q-characters

- More formally, we can introduce the operator:

$$\mathcal{K}^0 = \exp\left(-(q-q^{-1})\sum_{i\in I_0}\sum_{k>0}\frac{k}{[k]_q}H_{i,k}\otimes\tilde{h}_{i,-k}z^k\right) \in \mathbf{U}^\imath\otimes U_q(\widetilde{\mathfrak{h}})[\![z]\!],$$

and define the **boundary q-character map** to be

$$\chi_q^b\colon \operatorname{Rep}\mathbf{U}^\imath \to U_q(\widetilde{\mathfrak{h}})[\![z]\!], \qquad [V]\mapsto \operatorname{Tr}_V((\pi_V\otimes 1)(\mathcal{K}^0)).$$

- We do not know if $\mathcal{K}^0$ is the abelian part of the universal K-matrix.

Two POV

- $\imath$**-quantum groups** - independent structures, generalizations of quantum groups
- **QSP** - compatibility between $\mathbf{U}^\imath$ and $U_q(\widehat{\mathfrak{g}})$

Compatibility

Taking the QSP point of view, the two q-character maps χ_q and χ_q^b should be **compatible**.

$U_q(L\mathfrak{g})$	$\mathbf{U}^\imath$
χ_q - ring homo	χ_q^b - module homo

Twisting action: $U_q(\widetilde{\mathfrak{h}})[\![z]\!]$ as a $\mathbb{Z}[Y_{i,a}^{\pm 1}]$-module via the ring homo

$$\mathbb{Z}[Y_{i,a}^{\pm 1}] \to \mathbb{Z}[Y_{i,a}^{\pm 1}] \hookrightarrow U_q(\widetilde{\mathfrak{h}})[\![z]\!], \qquad Y_{i,a} \mapsto \mathbf{Y}_{i,a} = Y_{i,a} Y_{\sigma(i),a^{-1}}^{-1}.$$

Compatibility theorem (P., Li–P.)

Our coproduct theorem implies that the diagram below commutes:

$$\begin{array}{ccc} [\mathrm{Rep}\, U_q(\widehat{\mathfrak{g}})] & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}] \\ \curvearrowright & & \curvearrowright \\ [\,\mathrm{Rep}\, \mathbf{U}^\imath] & \xrightarrow{\chi_q^b} & U_q(\widetilde{\mathfrak{h}})[\![z]\!], \end{array} \quad \text{i.e.,}$$

$$\chi_q^b(V \otimes W) = \chi_q^b(V) \cdot \chi_q(W).$$

Evaluation representations

Challenges

Consider $U_q(L\mathfrak{sl}_n)$:

- Can approach rep theory from both Drinfeld and FRT points of view.
- Usual methods: CSA, triangular decomposition, weight decomposition, HW/Verma theory.
- Gelfand–Tsetlin bases are weight bases.

In contrast, for $\imath$quantum groups or twisted Yangians:

- No obvious CSA, weight concept, or triangular decomposition.
- Some classification results from the FRT (reflection equation) point of view (Molev, Gow–Molev).
- Problem: Gauss decomposition does not appear to translate that HW concept into anything reasonable on the Drinfeld side.

FRT	Drinfeld
classification	q-character theory

Ansatz: Look at evaluation representations in type AI.
Advantage: $U_q'(\mathfrak{so}_n)$ is well-understood without FRT.

Relation to Gelfand–Tsetlin theory

- Recall $\langle K_i^{\pm 1} = \phi_{i,0}^{\pm} \rangle \subset \langle \phi_{i,r}^{\pm} \rangle$, and so χ_q is a refinement of the usual character map

$$\begin{array}{ccc} [\operatorname{Rep} U_q(L\mathfrak{sl}_n)] & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}] \\ \downarrow{\scriptstyle res} & & \downarrow \\ [\operatorname{Rep} U_q(\mathfrak{sl}_n)] & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}] \end{array}$$

  Gelfand–Tsetlin basis is a weight basis.

- What is χ_q^b lifting (Here $\Theta_{i,0} = 1$)?

$$\begin{array}{ccc} [\operatorname{Rep} Y_q^{tw}(\mathfrak{so}_n)] & \xrightarrow{\chi_q^b} & U_q(\widetilde{\mathfrak{h}})[\![z]\!] \\ \downarrow{\scriptstyle res} & & \downarrow \\ [\operatorname{Rep} U_q'(\mathfrak{so}_n)] & \xrightarrow{???} & ??? \end{array}$$

  Answer: A kind of **Gelfand–Tsetlin character**.

Nonstandard deformation of $\mathfrak{so}_n$

Definition

Recall the q-Serre relation:

$$\mathrm{Serre}_{ij}(x,y) = \begin{cases} x^2y - [2]xyx + yx^2 & \text{if } a_{ji} = -1, \\ xy - yx & \text{if } a_{ji} = 0. \end{cases}$$

Let $\mathbf{U}^\imath = U_q'(\mathfrak{so}_n)$ *be the algebra generated by* B_i *and central invertible elements* $\mathbb{K}_i$ *subject to relations*

$$-q\mathbb{K}_i^{-1}\mathrm{Serre}_{ij}(B_i, B_j) = \begin{cases} 0 & \text{if} \quad a_{ji} = 0, \\ B_j & \text{if} \quad a_{ji} = -1. \end{cases}$$

FD representations

Approach 1: Verma theory (Wenzl)

- $\langle B_1, B_3, B_5, \ldots \rangle$ - quantizes a Cartan subalgebra of $\mathfrak{so}_n$,
- HW condition:

$$B_{2i-1} v_0 = [\lambda_i] v_0, \qquad B_{2i-1} B_{2i} v_0 = [\lambda_i - 1] B_{2i} v_0.$$

Approach 2: Gelfand–Tsetlin theory (Gavrilik–Iorgov–Klimyk)

- Consider restrictions wrt the chain of inclusions

$$\cdots \subset U_q'(\mathfrak{so}_{n-2}) \subset U_q'(\mathfrak{so}_{n-1}) \subset U_q'(\mathfrak{so}_n).$$

  These are multiplicity-free and, hence, yield a canonical (Gelfand–Tsetlin) basis.
- Gelfand–Tsetlin basis is **not** a weight basis.

Classification of FD simples:

- Classical modules - quantizations of simple $U(\mathfrak{so}_n)$-modules
- Non-classical modules - no well-defined classical limit

Gelfand–Tsetlin basis

Basis (of a FD simple module) is parametrised by GT patterns σ:

$$
\begin{array}{ccccccccc}
m_{1,2k+1} & & m_{2,2k+1} & & \cdots & & m_{k,2k+1} & & \\
& m_{1,2k} & & m_{2,2k} & & & & & |m_{k,2k}| \\
& & \cdots & & & & & & \\
& & & \cdots & & & & & \\
& & & & m_{1,5} & & m_{2,5} & & \\
& & & & & m_{1,4} & & & |m_{2,4}| \\
& & & & & & m_{1,3} & & \\
& & & & & & & & |m_{1,2}|
\end{array}
$$

The action of $U_q'(\mathfrak{so}_n)$ is given by GT-type formulae:

$$B_{2p} \cdot \sigma = \sum_{j=1}^{p} P_{2p}^{j}(\sigma)\sigma_{+j}^{2p} - \sum_{j=1}^{p} P_{2p}^{j}(\sigma_{-j}^{2p})\sigma_{-j}^{2p},$$

$$B_{2p-1} \cdot \sigma = \sum_{j=1}^{p-1} Q_{2p-1}^{j}(\sigma)\sigma_{+j}^{2p-1} - \sum_{j=1}^{p-1} Q_{2p-1}^{j}(\sigma_{-j}^{2p-1})\sigma_{-j}^{2p-1} - R_{2p-1}(\sigma)\sigma.$$

Evaluation homomorphism

- There are surjective maps (depending on $a \in \mathbb{C}^*$)

$$\mathrm{ev}_a \colon Y_q^{tw}(\mathfrak{so}_n) \twoheadrightarrow U'(\mathfrak{so}_n).$$

- First described by Molev–Sorba–Ragoucy in the FRT presentation.
- The evaluation map can also be described more explicitly in terms of iterated q-brackets:

$$\mathrm{ev}_a(B_0) = a\mathbb{K}_1^{-1}\cdots\mathbb{K}_n^{-1}[B_1,[B_2,\cdots,[B_{n-1},B_n]_q\cdots]_q]_q,$$
$$\mathrm{ev}_a(\mathbb{K}_0) = a^2\mathbb{K}_1^{-1}\cdots\mathbb{K}_n^{-1}.$$

Main results

Inspiration

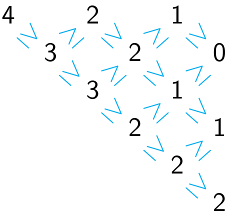
- Frenkel–Mukhin: *The Hopf algebra* Rep $U_q\widehat{\mathfrak{gl}}_\infty$
- Nazarov–Tarasov: *Representations of Yangians with Gelfand–Zetlin bases*

Theorem (Li-P.)

1. We calculate the action of the Drinfeld generators on the GT basis.
2. The GT basis is an **eigenbasis** for the operators $\Theta_{i,k}$.
3. We obtain explicit formulas for the boundary q-characters of evaluation modules.

Sketch of the combinatorial formula

- To a GT pattern, one associates a semistandard Young tableau:



$\longleftrightarrow$

1	1	4	6
3	5		
6			

- To a tableau, we associate the monomial ($\Upsilon_{i,b} = \mathbf{Y}_{i,bq^i}^{-1}\mathbf{Y}_{i+1,bq^{i-1}}$)

$$\Xi(T) = \prod_{(i,j)\in T} \Upsilon_{t(i,j),q^{2c(i,j)}} = \Upsilon_{1,1}\Upsilon_{1,q^2}\Upsilon_{3,q^{-2}}\Upsilon_{4,q^4}\Upsilon_{5,1},\Upsilon_{6,q^{-4}}\Upsilon_{6,q^6}.$$

- Sum over all GT patterns with top row $\boldsymbol{\lambda}$

$$\chi_q^b(L_{\boldsymbol{\lambda}}(a)) = \sum_{T\in \mathrm{SST}(\boldsymbol{\lambda})} 2^{\gamma_T}\cdot \Xi(T).$$

Examples: first fundamental rep

The module $L_{(1,0)}(a)$ of $U_q'(\mathfrak{so}_5)$ has the GT basis:

$$
\begin{array}{cccc}
1 & & 0 & \\
 & 1 & & 0 \\
 & & 1 & \\
 & & & 1
\end{array}
\qquad
\begin{array}{cccc}
1 & & 0 & \\
 & 1 & & 0 \\
 & & 1 & \\
 & & & -1
\end{array}
\qquad
\begin{array}{cccc}
1 & & 0 & \\
 & 1 & & 0 \\
 & & 0 & \\
 & & & 0
\end{array}
$$

$$
\begin{array}{cccc}
1 & & 0 & \\
 & 0 & & 0 \\
 & & 0 & \\
 & & & 0
\end{array}
\qquad
\begin{array}{cccc}
1 & & 0 & \\
 & 1 & & 0 \\
 & & 1 & \\
 & & & 0
\end{array}
$$

Examples and symmetries of boundary q-characters

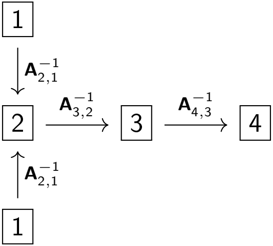
The boundary q-character for the module $L_{(1,0)}(a)$ is

$$\begin{array}{ccccc} \mathbf{Y}_{1,1}^{-1} & & & & \\ \big\downarrow \mathbf{A}_{2,1}^{-1} & & & & \\ \mathbf{Y}_{2,2}^{-1}\mathbf{Y}_{3,1} & \xrightarrow{\mathbf{A}_{3,2}^{-1}} & \mathbf{Y}_{3,3}^{-1}\mathbf{Y}_{4,2} & \xrightarrow{\mathbf{A}_{4,3}^{-1}} & \mathbf{Y}_{4,4}^{-1} \\ \big\uparrow \mathbf{A}_{2,1}^{-1} & & & & \\ \mathbf{Y}_{1,1}^{-1} & & & & \end{array}$$

where we write $\mathbf{Y}_{i,s} = Y_{i,aq^s} Y_{i,aq^{-s}}^{-1}$, $\mathbf{A}_{i,s} = A_{i,aq^s} A_{i,aq^{-s}}^{-1}$. Note that $\mathbf{Y}_{1,0} = \mathbf{Y}_{2,0} = 1$, $\mathbf{Y}_{1,1}^{-1} = \mathbf{Y}_{1,-1}$.

Boundary q-character formula using tableaux

The above example of boundary q-character in terms of tableaux is



Thank you!!!