Algebras Generated By The Bott-Chern Forms On Flag Varieties And Graphs

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Graphical Zonotopal Algebras

Let G be a graph on vertices $V(G) = \{0\} \cup [n]$. For an integer k, consider the ideal $J_G^{(k)}$ in the ring $\mathbb{R}[x_1, \cdots, x_n]$ generated by

$$p_I^{(k)} := \left(\sum_{i \in I} x_i\right)^{d_I + k}, \quad I \subseteq [n],$$

where d_I is the number of edges of G connecting I and $V(G) \setminus I$.

$$\mathcal{C}_G^{(k)} := \mathbb{R}[x_1, \ldots, x_n] / J_G^{(k)}.$$

Remark

These algebras are independent on choice of the root.

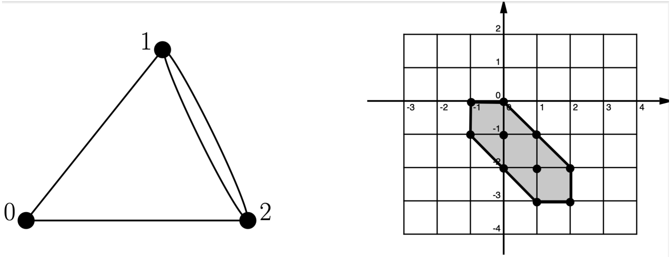
- $k = 1$: External Zonotopal algebra;
- $k = 0$: Central Zonotopal algebra;
- $k = -1$: Internal Zonotopal algebra.

Theorem (Postnikov-Shapiro; Ardila-Postnikov; Holtz-Ron)

Given a connected graph G. The Hilbert series of Zonotopal algebras are given by

- $\mathcal{H}_{\mathcal{C}_G^{(1)}}(q) = q^{e(G)-v(G)+1} T_G(1+q, \frac{1}{q})$;
- $\mathcal{H}_{\mathcal{C}_G^{(0)}}(q) = q^{e(G)-v(G)+1} T_G(1, \frac{1}{q})$;
- $\mathcal{H}_{\mathcal{C}_G^{(-1)}}(q) = q^{e(G)-v(G)+1} T_G(0, \frac{1}{q})$,

where T_G is the Tutte polynomial of G.



In this case zonotopal ideals are given by the formula

$$\mathcal{J}_G^{(k)} = \langle x_1^{3+k}, x_2^{3+k}, (x_1+x_2)^{2+k} \rangle.$$

- $\mathcal{H}_{\mathcal{C}_G^{(1)}}(q) = 1 + 2q + 3q^2 + 3q^3 + q^5$ and the total dimension $\dim(\mathcal{C}_G^{(1)}) = 10$;
- $\mathcal{H}_{\mathcal{C}_G^{(0)}}(q) = 1 + 2q + 2q^2$ and the total dimension $\dim(\mathcal{C}_G^{(0)}) = 5$;
- $\mathcal{H}_{\mathcal{C}_G^{(-1)}}(q) = 1 + q$ and the total dimension $\dim(\mathcal{C}_G^{(-1)}) = 2$.

Theorem (N.)

For graphs G_1, G_2*, the external zonotopal algebras* $\mathcal{C}^{(1)}_{G_1}$ *and* $\mathcal{C}^{(1)}_{G_2}$ *are isomorphic if and only if graphical matroids* M_{G_1} *and* M_{G_2} *are isomorphic.*

Conjecture (N.)

For graphs G_1, G_2, the central zonotopal algebras $\mathcal{C}^{(0)}_{G_1}$ and $\mathcal{C}^{(0)}_{G_2}$ are isomorphic if and only if graphical matroids $M_{G'_1}$ and $M_{G'_2}$ are isomorphic, where G'_i is the graph obtained from G_i after removal all bridges.

Theorem (Eur-Huh-Larson)

The Hilbert series of the external zonotopal algebras are log-concave.

Let $\Phi_G^{(1)}$ be the graded commutative algebra over $\mathbb{C}$ generated by the variables $\phi_e, e \in G$, with the defining relations:

$$(\phi_e)^2 = 0, \quad \text{for every edge } e \in G.$$

Theorem (Postnikov-Shapiro-Shapiro)

For any graph G, $\mathcal{C}_G^{(1)}$ is isomorphic to the subalgebra of $\Phi_G^{(1)}$ generated by the elements

$$X_i = \sum_{e \in G} \pm \phi_e.$$

Fomin-Kirillov

Fomin-Kirilov algebra $\mathcal{FK}_n$ is generated by $\phi_{ij}, i \neq j \in [n]$ with relations

- $\phi_{ij}^2 = 0$;
- $\phi_{ij} = -\phi_{ji}$;
- $\phi_{ij}\phi_{jk} + \phi_{jk}\phi_{ki} + \phi_{ki}\phi_{ij} = 0$;
- $\phi_{ij}\phi_{kl} = \phi_{kl}\phi_{ij}$ for $\{i,j\} \cap \{k,l\} = \emptyset$.

Theorem (Fomin-Kirillov)

The subalgebra of $\mathcal{FK}_n$ generated by $\theta_j = \sum_{i<j} \phi_{ij} - \sum_{k>j} \phi_{jk}, j \in [n]$ is commutative.

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Conjecture (Fomin-Kirillov)

For any permutation $w \in S_n$, $\mathfrak{S}_w(\theta_1, \theta_2, \ldots, \theta_{n-1})$ can be written as a positive sum of $\phi_{i_1 j_1}\phi_{i_2 j_2} \ldots \phi_{i_{\ell(w)} j_{\ell(w)}}$.

Part 2

Grassmannian and Incomplete grassmanian algebra

Let D be a diagram of boxes in $d \times (n-d)$, where columns are indexed by $1, 2, \ldots$ and rows are indexed by $\bar{1}, \bar{2}, \ldots$.

Define a bipartite directed graph $\mathcal{G}_D$ as follows

- The two sets of vertices are columns and rows of $d \times (n-d)$. (indexed by $\{1, 2 \ldots, d\} \subset \mathbb{N}$ and $\{\bar{1}, \bar{2}, \ldots, \overline{n-d}\} \subset \bar{\mathbb{N}}$);
- For any box $(i, j) \in D$ we have exactly two edges $\overrightarrow{(i, \bar{j})}$ and $\overrightarrow{(\bar{j}, i)}$.
- We are working with the exterior algebra of edges.

A $2d$-th Chern form is

$$\overleftarrow{c}_{2d} = \sum_{B \in \overleftarrow{\mathcal{E}}_D,\ |B|=2d} cp(B)\overleftarrow{m}(B),$$

where

- $\overleftarrow{\mathcal{E}}_D = \{B \text{ is Eulerian}:\ \forall i \in [d]\ indeg_B(i) = 0 \text{ or } 1\}$.
- $cp(B) = \prod_{j \in [n-d]} indeg_B(\bar{j})!$ is the number of cycle partitions of B.
- $\overleftarrow{m}(B)$ is the product of edges of B in some order.

Proposition (N.-Postnikov-Shapiro-Shapiro)

$$1 + \overleftarrow{c}_2 t + \overleftarrow{c}_4 t^2 + \ldots = \prod (1 + \overleftarrow{m}(C) t^{|C|/2}),$$

where the product is over all directed cycles of $\mathcal{G}_D$.

Corollary

$$(1 + \overleftarrow{c}_2 t + \overleftarrow{c}_4 t^2 + \ldots)(1 + \vec{c}_2 t + \vec{c}_4 t^2 + \ldots) = 1.$$

Let $\mathcal{FPS}_D$ be the algebra generated by Chern forms of D.

Theorem (N.-Postnikov-Shapiro-Shapiro)

Given a Young diagram D. The linear dimension of the k-graded component of $\mathcal{FPS}_D$ is equal to the number of Young diagrams inside D with exactly k boxes.

Problem

c_D^{2d} corresponds to h_d. What are Schur functions?

Flag varieties and graphs

Let G be a graph on k labeled vertices. And $a_i,\ i \in [k]$ are integer positive numbers.

- Let $n = a_1 + \ldots + a_k$. We will consider a partition $A_1 \sqcup A_2 \sqcup \ldots \sqcup A_k = [n]$, such that $|A_i| = a_i$.
- Let $\widetilde{G}$ be the following directed graph:
    - $V(G) = [n]$;
    - For every edge $(i, j) \in G$, we have $2a_ia_j$ its oriented clones, which form $\overleftrightarrow{K}_{A_i, A_j}$;
- We consider the exterior algebra of edges of $\widetilde{G}$.

- $\mathcal{FPS}_{G,a_1,\ldots,a_k}$ is generated by

$$c_{2d}^{(i)} = \sum_{B\in\mathcal{E}_i,\ |B|=2d} cp(B)m_i(B),$$

where $\mathcal{E}_i$ is the set of eulerian graphs in $\widetilde{G}$ such that
  - for any edge, one end is in A_i;
  - indegree of any vertex from A_i is 0 or 1;

and
  - $cp(B)$ is the number of cycle partitions of B, i.e.,

$$cp(B) = \prod_{j\in[n]} indeg_B(j)!\,;$$

  - $m_i(B)$ is the product of edges of B in some special order.

- $\mathcal{FPS}_{G,1,\dots,1}$ is the external Graphical Zonotopal algebra of G.
- $\mathcal{FPS}_{K_k,a_1,\dots,a_k}$, $a_i \in \mathbb{N}$ is the algebra corresponding to the flag variety

$$(0;\ a_1;\ a_1 + a_2;\ \dots;\ \sum_{i=1}^{t} a_i;\ \dots;\ \sum_{i=1}^{k-1} a_i;\ n)$$

In particular, $\mathcal{FPS}_{K_2,m,n-m}$ is isomorphic to the Cohomology ring of Grassmanian.

$$\prod_{\vec{e}\in\widetilde{G},\ e_1<e_2}(x_{e_1}+x_{e_2})=\sum_{\lambda_1,\dots,\lambda_k}c(\lambda_1,\dots,\lambda_k)\prod_{i=1}^{k}s_{\lambda_i}(x_j:\ j\in A_i),$$

where edge $\vec{e}=(e_1,e_2)$ and λ_i are Young diagrams.

Conjecture (N.-Postnikov-Shapiro-Shapiro)

The dimension of $\mathcal{FPS}_{G,a_1,\dots,a_k},\ a_i\in\mathbb{N}$ is equal to the number of non-zero coefficients $c(\lambda_1,\dots,\lambda_k)$.

Part 3

For the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, \ldots]$, the *i*-th divided differences operator is given by

$$\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}.$$

Definition (Lascoux–Schützenberger)/
Theorem (Demazure and Bernstein–Gelfand–Gelfand)

For a permutation $w_0 = (n, n-1, \ldots, 1) \in S_n$, we define its Schubert polynomial as

$$\mathfrak{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}^{1} \in \mathbb{Q}[x_1, x_2, \ldots].$$

For a permutation $w \in S_n$,

$$\mathfrak{S}_w := \partial_i \mathfrak{S}_{ws_i} \text{ if } \ell(ws_i) = \ell(w) + 1.$$

Theorem (Lascoux–Schützenberger)

For any $u \in S_{\mathbb{N}}$, *its Schubert polynomial* $\mathfrak{S}_u$ *is well defined and* $\mathfrak{S}_u$ *is a homogeneous polynomial of degree* $\ell(u)$.
The set $\{\mathfrak{S}_u, u \in S_{\mathbb{N}}\}$ *of all Schubert Polynomials forms a linear basis of* $\mathbb{Q}[x_1, x_2, x_3, \ldots]$.

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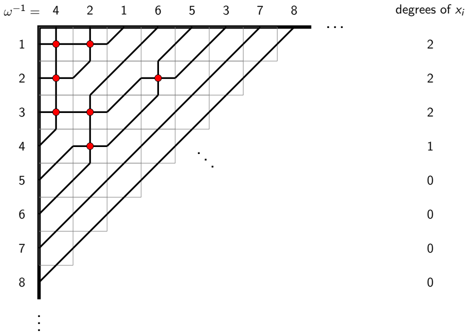
Therefore, we have unique coefficients $c_{u,v}^{w}, (u, v, w \in S_{\mathbb{N}})$ such that, for any $u, v \in S_{\mathbb{N}}$,

$$\mathfrak{S}_u \mathfrak{S}_v = \sum_{w \in S_{\mathbb{N}}} c_{u,v}^{w} \mathfrak{S}_w.$$

Problem

Give a combinatorial interpretation of $c_{u,v}^{w}$.

RC graphs/ Pipe dream



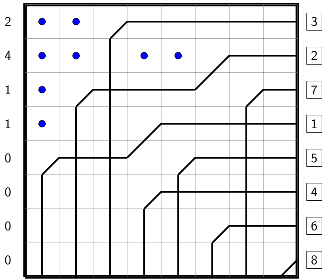
$\boxed{m = x_1^2x_2^2x_3^2x_4}$

Proposition (Fomin-Kirillov)

For any permutation $w \in S_{\mathbb{N}}$, its Schubert polynomial is given by

$$\mathfrak{S}_w = \sum_{g \in \mathcal{RC}(w)} m(g).$$

Bumpless Pipe dream



$\boxed{m(g) = x_1^2 x_2^4 x_3 x_4}$

Proposition (Lam-Lee-Shimozono)

For any permutation $w \in S_{\mathbb{N}}$, its Schubert polynomial is given by

$$\mathfrak{S}_w = \sum_{g \in BPD(w)} m(g).$$

Monk's rule (Monk)

For $u \in S_{\mathbb{N}}$ and $m \in \mathbb{N}$, we have

$$\mathfrak{S}_u \mathfrak{S}_{s_m} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots + x_m) = \sum_{a \leq m < b:\ \ell(ut_{a,b}) = \ell(u)+1} \mathfrak{S}_{ut_{a,b}},$$

where $t_{a,b}$ is a transposition of a and b.

Monk's rule (Monk)

For diagram λ, we have

$$s_\lambda s_{\square} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots) = \sum_{\mu} s_\mu,$$

where summation is over all Young diagrams μ s.t. μ/λ is one box.

Let $[a, b], a < b \in \mathbb{N}$ be

$$\mathfrak{S}_u[a, b] = \begin{cases} \mathfrak{S}_{ut_{a,b}} & \text{if } \ell(ut_{a,b}) = \ell(u) + 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathfrak{S}_u \mathfrak{S}_{s_m} = \mathfrak{S}_u \cdot (x_1 + x_2 + \ldots + x_m) = \sum_{a \leq m < b} \mathfrak{S}_u[a, b].$$

Remark

Operators $\{[i, j], 1 \leq i < j \leq n\}$ satisfies the relations of Fomin-Kirillov algebra.

Pieri's rule (Sottile)

For $u \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, we have

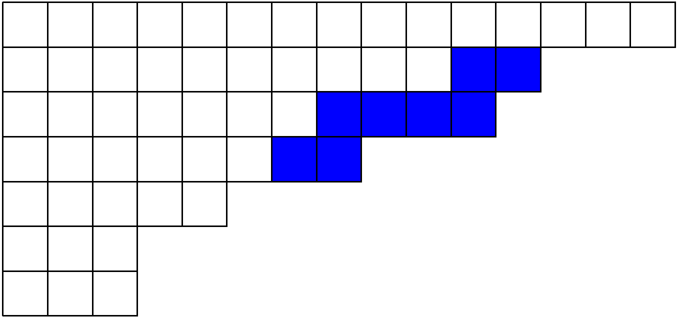
$$\begin{aligned}\mathfrak{S}_u \cdot h_k(x_1, x_2, \dots, x_m) &= \mathfrak{S}_u \cdot \left(\sum_{i_1 \le i_2 \le \dots \le i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k} \right) = \\ &= \sum_{\substack{a_1 \le \dots \le a_k \le m \\ m < b_1, \dots, b_k \text{ are distinct}}} \mathfrak{S}_u [a_1 b_1][a_2 b_2] \cdots [a_k b_k]\end{aligned}$$

and

$$\begin{aligned}\mathfrak{S}_u \cdot e_k(x_1, x_2, \dots, x_m) &= \mathfrak{S}_u \cdot \left(\sum_{i_1 < i_2 < \dots < i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k} \right) = \\ &= \sum_{\substack{a_1, \dots, a_k \le m \text{ are distinct} \\ m < b_1 \le \dots \le b_k}} \mathfrak{S}_u [a_1 b_1][a_2 b_2] \cdots [a_k b_k]\end{aligned}$$

Theorem (Murnaghan-Nakayama)

$$s_\lambda p_k = \sum_{\substack{\mu:\ \lambda\subset\mu,\ |\mu|=|\lambda|+k,\\ \mu\backslash\lambda \textit{ is a border strip}}} (-1)^{ht(\mu\backslash\lambda)} s_\mu.$$

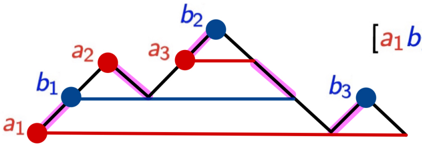


Murnaghan–Nakayama rule (N.)

For $u \in S_{\mathbb{N}}$ and $k, m \in \mathbb{N}$, we have

$$\mathfrak{S}_u \cdot p_k(x_1, x_2, \ldots, x_m) = \mathfrak{S}_u \cdot (x_1^k + x_2^k + \ldots + x_m^k) =$$

$$= \sum_{P \text{ is a Dyck path of length } 2k} (-1)^{u_e(P)} \sum_{\substack{a_1,\ldots,a_{u_e(P)+1} \leq m \\ b_1,\ldots,b_{d-u_e(P)} > m \\ \text{are distinct}}} \mathfrak{S}_u \mathcal{M}_P(a, b).$$



$$[a_1 b_1][a_2 b_1][a_3 b_2][a_3 b_1][a_1 b_3]$$

Thank You!