Tropical symmetries of cluster algebras

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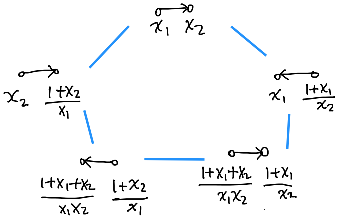
Cluster algebras

- Cluster algebras are a class of commutative algebras introduced by Fomin and Zelevinsky.
- Each cluster algebra is defined using some initial data called a seed and using a procedure called mutation.

Example (cluster algebra of type A_2)

initial seed $\quad \underset{x_1}{\circ} \longrightarrow \underset{x_2}{\circ} \qquad (x, Q),\ x = (x_1, x_2),\ Q = \circ \longrightarrow \circ$



$$A(x, Q) = \mathbb{C}\left[x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right].$$

mutate at vertex k, $\mu_k(Q)$ is obtained from Q by:

(1) for every $i \to k \to j$, add $i \longrightarrow j$.

(2) change

(3) erase every $\rightleftarrows$.

$(x_1, \cdots, x_n)$ is changed to $(x_1, \cdots, x_k', \cdots, x_n)$, $x_k' = \frac{1}{x_k}\left(\prod_{i \to k} x_i + \prod_{k \to j} x_j\right)$.

Grassmannian cluster algebras

- Let $k \leq n \in \mathbb{Z}_{\geq 1}$ and

$$\begin{aligned}\mathrm{Gr}(k, n) &= \{k \text{ dimensional subspaces of } \mathbb{C}^n\} \\ &= \{k \times n \text{ full rank matrices}\}/\text{row operations.}\end{aligned}$$

- Scott (arXiv:math/0311148) showed that the coordinate ring $\mathbb{C}[\mathrm{Gr}(k, n)]$ has a cluster algebra structure.
- The algebra $\mathbb{C}[\mathrm{Gr}(k, n)]$ is called a Grassmannian cluster algebra.

Grassmannian cluster algebras and physics

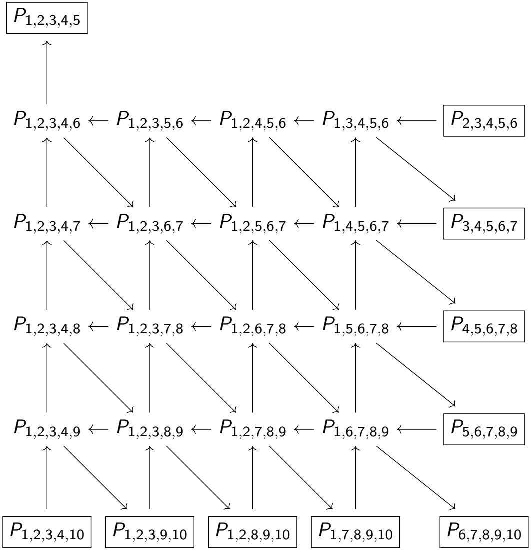
Recently, Grassmannian cluster algebras are found to be an important tool in scattering amplitudes in physics, see for examples,

- Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka, Scattering amplitudes and the positive Grassmannian, (2012).
- Arkani-Hamed, Lam, Spradlin, J. High Energ. Phys. 65 (2021),
- Arkani-Hamed, He, Lam, SIGMA 17 (2021), 092, 41 pages.
- He, Li, Yang, J. High Energ. Phys. 119 (2021).
- Chicherin, Henn, Papathanasiou, Phys. Rev. Lett. 126 (2021) 9, 091603,
- Drummond, Foster, Gürdoğan, Kalousios, J. High Energ. Phys. 146 (2020),
- Golden, Paulos, Spradlin, Volovich, J. Phys. A: Math. Theor. 47 474005,
- Henke, Papathanasiou, J. High Energ. Phys. 2021, 7 (2021).

An initial seed for a Grassmannian cluster algebra

- There is a seed of the Grassmannian cluster algebra $\mathbb{C}[\mathrm{Gr}(k,n)]$ whose cluster variables are given by Plücker coordinates, [arXiv:math/0311148, Scott].
- A Plücker coordinate $P_{i_1,\ldots,i_k} \in \mathbb{C}[\mathrm{Gr}(k,n)]$ ($i_1 < \cdots < i_k$): for a $k \times n$ matrix $x = (x_{ij})_{k\times n}$, $P_{i_1,\ldots,i_k}(x)$ is the minor of x with 1st, ..., kth rows and i_1th, ..., i_kth columns.

An initial cluster for $\mathbb{C}[\mathrm{Gr}(5,10)]$



An example of exchange relation

- The relation

$$P_{13456}P_{24567} = P_{14567}P_{23456} + P_{12456}P_{34567}$$

  is an example of exchange relation (corresponding to mutation at the vertex P_{13456} in the initial quiver). It is a Plücker relation.
- In general, exchange relations are more complicated than Plücker relations.

Monoid $\mathrm{SSYT}(k, [n])$ of semi-standard Young tableaux

- $\mathbb{C}[\mathrm{Gr}(k,n)]$ has a nice basis called dual canonical basis which contains all cluster monomials. The dual canonical basis elements are in one to one correspondence with semistandard tableaux.
- $\mathrm{SSYT(k, [n])}$ = the set consisting of the empty tableau and semi-standard Young tableaux of rectangular shape with k rows and with entries in $[n]$.
- For $A, B \in \mathrm{SSYT(k, [n])}$, $A \cup B$ is the semi-standard tableau with k rows and the elements in the ith row are the union of elements in the ith row of A and B, $i \in [k]$.

Example

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 7 \\ \hline 6 & 8 \\ \hline \end{array} \cup \begin{array}{|c|c|}\hline 1 & 7 \\ \hline 2 & 9 \\ \hline 8 & 10 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 7 \\ \hline 2 & 2 & 7 & 9 \\ \hline 6 & 8 & 8 & 10 \\ \hline \end{array}.$$

Monoid $\mathrm{SSYT}(k,[n])$ of semi-standard Young tableaux

- For $A, B \in \mathrm{SSYT}(k,[n])$, define $A \sim B$ if either A, B are trivial tableaux or $\mathrm{red}(A) = \mathrm{red}(B)$.

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} \sim \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline 4 & 5 & 6 \\ \hline \end{array} = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 6 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \cup \begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline 5 \\ \hline \end{array}.$$

- Denote $\mathrm{SSYT}(k,[n],\sim) = \mathrm{SSYT}(k,[n])/\sim$.

Lemma

$\mathrm{SSYT}(k,[n])$ *and* $\mathrm{SSYT}(k,[n],\sim)$ *are commutative cancellative monoids under the multiplication "$\cup$".*

Dual canonical basis of a Grassmannian cluster algebra

Define $\mathbb{C}[\mathrm{Gr}(k,n,\sim)] = \mathbb{C}[\mathrm{Gr}(k,n)]/\langle P_{i,i+1,\ldots,i+k-1} - 1, i \in [n-k+1]\rangle$.

Theorem (Chang-Duan-Fraser-L. 2020)

Every element in the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n,\sim)]$ is of the form $\widetilde{\mathrm{ch}}(T)$ for some tableau $T \in \mathrm{SSYT}(k,[n],\sim)$.

Moreover, every dual canonical basis element in $\mathbb{C}[\mathrm{Gr}(k,n)]$ is of the from $\mathrm{ch}(T)$ for some tableau in $T \in \mathrm{SSYT}(k,[n])$ [Drummond-Gürdoğan-L. 2026].

There are explicit formulas for $\widetilde{\mathrm{ch}}(T)$ and $\mathrm{ch}(T)$ [Chang-Duan-Fraser-L. 2020].

A formula for $\mathrm{ch}(T)$

The formula $\widetilde{\mathrm{ch}}(T) = \sum_{u \in S_r} (-1)^{\ell(uw_T)} p_{uw_0, w_T w_0}(1) P_{u;T'}$ is obtained as follows.
[Arakawa-Suzuki 98, see also Henderson 07, Barbasch-Ciubotaru 15, Lapid-Minguez 18] For a multisegment $\mathbf{m}$ with r terms,

$$[\mathrm{Z}(\mathbf{m})] = [\mathrm{Z}(\mathbf{m}_{w_{\mathbf{m}} \cdot \mu_{\mathbf{m}}, \lambda_{\mathbf{m}}})] = \sum_{w' \in S_r} (-1)^{\ell(w' w_{\mathbf{m}})} p_{w' w_0, w_{\mathbf{m}} w_0}(1) [\zeta(\mathbf{m}_{w' \cdot \mu_{\mathbf{m}}, \lambda_{\mathbf{m}}})].$$

We [CDFL20] translate this formula to a formula of q-characters [Frenkel–Reshetikhin 98] using quantum affine Schur-Weyl duality [Chari–Pressley 97]

$$\chi_q(L(M)) = \sum_{w' \in S_r} (-1)^{\ell(w' w_M)} p_{w' w_0, w_M w_0}(1) \prod_{M' \in \mathrm{Fund}_M(w' \mu_M, \lambda_M)} \chi_q(L(M')).$$

The formula of $\widetilde{\mathrm{ch}}(T)$ is obtained from the above formula using an isomorphism $K_0(\mathcal{C}_\ell) \cong \mathbb{C}[\mathrm{Gr}(k, n, \sim)]$ $(n = k + \ell + 1)$ [Hernandez–Leclerc 2010].

A formula for ch(T)

- We say that a tableau T is a fundamental tableau if T has one column and the entries of T are $\{i, i+1, \ldots, j, j+2, \ldots, k+i-1\}$ for some $i \leq j \in [n]$.
- Let $T = \begin{array}{|c|c|c|c|} \hline i_1 & i_2 & \cdots & i_r \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline \end{array}$ be a tableau with r columns and every column of T is a fundamental tableau.
- Define $i_T = (i_1, \ldots, i_r)$, $l_T = (l_1, \ldots, l_r)$, where l_a is defined by $[i_a, i_a + k] \setminus \{l_a\}$ is the set of entries of the ath column of T.
- Define $j_T = (j_1, \ldots, j_r)$ to be the reordering of $l_1, \ldots, l_r$ such that $j_1 \leq \ldots \leq j_r$.
- For a tableau T, define $P_T = P_{T_1} \cdots P_{T_r}$, where $T_1, \ldots, T_r$ are columns of T. P_T is called the standard monomial of T.

A formula for ch(T)

- Let T be a tableau with r columns and each column is a fundamental tableau. For $u \in S_r$, we define $P_{u;T}$ as follows.
- If $j_a \in [i_{u(a)}, i_{u(a)} + k]$ for all $a \in [r]$, then define a tableau $\alpha(u; T)$ to be the semi-standard tableau whose columns have entries $[i_{u(a)}, i_{u(a)} + k] \setminus \{j_a\}$, $a \in [r]$ and define $P_{u;T} = P_{\alpha(u;T)}$. Otherwise define $P_{u;T} = 0$.
- There exists a unique element $w_T \in S_r$ with maximal length such that $P_{w_T;T} = P_T$.
- For any tableau $T \in \mathrm{SSYT}(k, [n])$, there is a unique tableau $T' \in \mathrm{SSYT}(k, [n])$ whose columns are fundamental tableaux such that $T \sim T'$. Define $w_T = w_{T'}$.

A formula for ch(T)

- For $T \in \mathrm{SSYT}(k, [n])$,

$$\widetilde{\mathrm{ch}}(T) = \sum_{u \in S_r} (-1)^{\ell(uw_T)} p_{uw_0, w_T w_0}(1) P_{u;T'} \in \mathbb{C}[\mathrm{Gr}(k,n,\sim)]$$

  where $T' \in \mathrm{SSYT}(k, [n])$ is the unique tableau whose columns are fundamental tableaux such that $T \sim T'$, $p_{x,y}(t)$ is the Kazhdan–Lusztig polynomial, and r is the number of columns of T'. Let $T'' = T' \cup T^{-1}$. Then $\mathrm{ch}(T) = \frac{1}{P_{T''}} \widetilde{\mathrm{ch}}(T)$, where $P_{T''} = P_{T''_1} \cdots P_{T''_p}$, T''_j's are columns of T''. We proved in [Drummond-Gürdoğan-L. 2026] that $\mathrm{ch}(T)$ is a polynomial.
- For example,

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 6 \\ \hline \end{array}\right) = P_{124}P_{356} - P_{123}P_{456},$$

$$\mathrm{ch}\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}\right) = P_{145}P_{236} - P_{123}P_{456}.$$

Quasi-homomorphisms of cluster algebras

- Chris Fraser 2017 introduced the concept of quasi-homomorphisms of cluster algebras.
- An algebra homomorphism $f : \mathcal{A} \to \mathcal{A}'$ is called a quasi-homomorphism if $f(\mathbb{P}) \subset \mathbb{P}'$, and there is a labelled seed $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ for $\mathcal{A}$, a labelled seed $\Sigma_{t'} = (\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'})$ for $\mathcal{A}'$ such that

  1. $f(x_{i;t}) \propto x_{i;t'}$, $1 \leq i \leq n$,
  2. $f(\hat{y}_{i;t}) = \hat{y}_{i;t'}$, $1 \leq i \leq n$,
  3. $B_t = B_{t'}$,

  where $\hat{y}_{j;t} = y_{j;t} \prod_{i=1}^{n} x_{i;t}^{b_{ij}^t}$.
- By definition, a quasi-homomorphism sends cluster variables to cluster variables (after removing frozen factors), and sends a cluster to a cluster.

g-vectors

- g-vector is an integer vector that identifies a cluster variable with respect to an initial seed.
- The initial cluster variables have the standard unit vectors as their g-vectors, and the g-vectors of all other cluster variables are obtained by applying the mutation rule for g-vectors.
- g-vectors can also be defined for every element of the dual canonical basis of $\mathbb{C}[\mathrm{Gr}(k,n)]$.

Tropicalization of quasi-automorphisms of cluster algebras

- Consider the cluster algebra with the dual exchange matrix $B^\vee = (b^\vee_{ij})$ ($B^\vee = -B^T$) for each labelled seed and denote its $\hat{y}$ variables by $\hat{y}^\vee_j$. Define tropicalised coordinates $\tilde{y}^\vee_j = \mathrm{Trop}_+ \hat{y}^\vee_j$.

Theorem (Drummond-Gürdoğan-L. 2026)

Let $\mathcal{A}$ be a cluster algebra of geometric type with a skew-symmetrisable exchange matrix and let $f : \mathcal{A} \to \mathcal{A}$ be a quasi-automorphism. Let Σ_t and $\Sigma_{t'}$ be labelled seeds obeying properties (1), (2) and (3) in the definition of a quasi-homomorphism. Let x be a cluster variable and let μ_f be the sequence of mutations such that $\Sigma_{t'} = \mu_f \Sigma_t$. Then the map

$$(\tilde{y}^\vee_{1;t'}, \ldots, \tilde{y}^\vee_{n;t'}) \mapsto (\tilde{y}^\vee_{1;t}, \ldots, \tilde{y}^\vee_{n;t}) \tag{1}$$

sends the g-vector of x to the g-vector of $f(x)$.

Tropicalization of quasi-automorphisms of cluster algebras

- The tropicalization of a quasi-automorphism sends a g-vector to a g-vector.
- Tropicalization of quasi-automorphisms gives a convenient way to compute quasi-automorphisms.

Braid group actions

- Fraser 2020 defined a braid group action on $\mathbb{C}[\mathrm{Gr}(k,n)]$.
- Let $d = \gcd(k,n)$. For $i \in [1, d-1]$, the map $\sigma_i : \mathrm{Gr}(k,n) \to \mathrm{Gr}(k,n)$ is defined as follows.
- Divide an element $(v_1, \ldots, v_n)$ (columns) in $\mathrm{Gr}(k,n)$ into $\frac{n}{d}$ windows: $[v_{1+jd}, \ldots, v_{(j+1)d}]$, $j \in [0, \frac{n}{d} - 1]$.
- The map σ_i sends the first window to $[v_1, \ldots, v_{i-1}, v_{i+1}, w_1, v_{i+2}, \ldots, v_d]$, where

$$w_1 = \frac{\det(v_i, v_{i+2}, \ldots, v_{i+k})}{\det(v_{i+1}, v_{i+2}, \ldots, v_{i+k})} v_{i+1} - v_i.$$

- The ℓth window is defined by the same recipe by d-periodically augmenting indices.
- The pullbacks σ_i^* (also denoted by σ_i) is a quasi-automorphism on $\mathbb{C}[\mathrm{Gr}(k,n)]$.

Braid group actions

- For each i, tropicalization of σ_i gives a map sending a g-vector to a g-vector, and sending a semistandard Young tableau to a semistandard Young tableau.
- We choose the web matrix $W = (\mathrm{id}_k \mid M)$, where $M = (M_{ij})_{k\times(n-k)}$, and

$$M_{ij} = (-1)^{k+i} \sum_{0\leq\lambda_{k-i}\leq\cdots\leq\lambda_1\leq j-1} \prod_{b=1}^{k-i} \prod_{a=1}^{\lambda_b} \chi_{a,b},$$

[Speyer–Williams 05].

Tropicalization of braid group action of cluster algebras

- Consider the case of $\mathrm{Gr}(3,6)$.
- Evaluating cluster $\hat{y}$-coordinates in the initial seed on W, we obtain $(\chi_{11}, \chi_{12}, \chi_{21}, \chi_{22})$.
- Applying σ_1^{-1} to the web matrix W and evaluate the $\hat{y}$-variables of the initial seed on $\sigma_1^{-1}(W)$ we have

$$\sigma_1^{-1} : \begin{bmatrix} \chi_{1,1} \\ \chi_{1,2} \\ \chi_{2,1} \\ \chi_{2,2} \end{bmatrix} \mapsto \begin{bmatrix} \frac{\chi_{2,2}}{1+\chi_{1,2}+\chi_{1,2}\chi_{2,2}} \\ \frac{1+\chi_{1,2}}{\chi_{1,2}\chi_{2,2}} \\ \chi_{1,1}(1+\chi_{1,2}) \\ \frac{\chi_{2,1}(1+\chi_{1,2}+\chi_{1,2}\chi_{2,2})}{1+\chi_{1,2}} \end{bmatrix}. \qquad (2)$$

- Tropicalising the above map gives the map $Q^+_{\sigma_1}$:

$$Q^+_{\sigma_1} : \begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{2,1} \\ v_{2,2} \end{bmatrix} \mapsto \begin{bmatrix} v_{2,2} - \max(0, v_{1,2}, v_{1,2} + v_{2,2}) \\ \max(0, v_{1,2}) - v_{1,2} - v_{2,2} \\ v_{1,1} + \max(0, v_{1,2}) \\ v_{2,1} + \max(0, v_{1,2}, v_{1,2} + v_{2,2}) - \max(0, v_{1,2}) \end{bmatrix} \tag{3}$$

For example, we have

$$Q^+_{\sigma_1}\mathbf{g}(P_{124}) = Q^+_{\sigma_1}(1,0,0,0) = (0,0,1,0) = \mathbf{g}(P_{125}). \tag{4}$$

Fixed points of quasi-automorphisms

- Let $\mathcal{A}$ be any cluster algebra of rank n and f is a quasi-automorphism on $\mathcal{A}$.
- We say that a g-vector $g \in \mathbb{Z}^n$ is a fixed point of f if $f(g) = g$.
- For a g-vector g which is fixed by a quasi-automorphism f on $\mathcal{A}$, we say that g is a stable fixed point if for every generic vector g' in $\mathbb{Z}^n$, the sequence $f^j(g')$, $j = 1, 2, \ldots$, has a limit g (meaning that $f^j(g')$ is very close to $c_j g$ for some $c_j \in \mathbb{Z}$ when j is very large).

Fixed points of the maps given by braid group generators

- Denote by ρ the cyclic shift map. It is a cluster automorphism on $\mathbb{C}[\mathrm{Gr}(k, n)]$.
- Denote $\sigma_3 = \rho \circ \sigma_2 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3 in $\mathbb{C}[\mathrm{Gr}(3, 9)]$ are:

1	3	4
2	6	7
5	8	9

,

1	2	5
3	4	8
6	7	9

,

1	2	3
4	5	6
7	8	9

.

- Denote $\sigma_4 = \rho \circ \sigma_3 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3, σ_4 in $\mathbb{C}[\mathrm{Gr}(4, 8)]$ are:

1	3
2	5
4	7
6	8

,

1	2
3	4
5	6
7	8

,

1	3
2	5
4	7
6	8

,

1	2
3	4
5	6
7	8

,

where the first and the third are the same, and the second and the fourth are the same.

Fixed points of the maps given by braid group generators

- Denote $\sigma_4 = \rho \circ \sigma_3 \circ \rho^{-1}$. The stable fixed points of σ_1, σ_2, σ_3, σ_4 in $\mathbb{C}[\mathrm{Gr}(4,12)]$ are:

1	3	5
2	7	9
4	8	11
6	10	12

,

1	2	6
3	4	8
5	9	10
7	11	12

,

1	3	7
2	5	9
4	6	11
8	10	12

,

1	2	4
3	6	8
5	7	10
9	11	12

.

- Denote $\sigma_5 = \rho \circ \sigma_4 \circ \rho^{-1}$. The stable fixed points of $\sigma_1, \ldots, \sigma_5$ in $\mathbb{C}[\mathrm{Gr}(5,10)]$ are:

1	3
2	6
4	8
5	9
7	10

,

1	2
3	4
5	7
6	9
8	10

,

1	3
2	5
4	6
7	8
9	10

,

1	4
2	6
3	7
5	9
8	10

,

1	2
3	5
4	7
6	8
9	10

.

- $\text{ch}(T) \in \mathbb{C}[\text{Gr}(k,n)]$ is called real if $\text{ch}(T)^2 = \text{ch}(T \cup T)$. $\text{ch}(T)$ is called prime if $\text{ch}(T) \neq \text{ch}(T')\text{ch}(T'')$ for any non-empty tableaux T', T''.
- We proved that the number of rank r prime non-real elements in the dual canonical basis of $\mathbb{C}[\text{Gr}(3,9)]$ which can be obtained by applying the braid group action to the following three tableaux: $\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&6&7\\\hline 5&8&9\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&8\\\hline 6&7&9\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7&8&9\\\hline\end{array}$ is $3\phi(\frac{r}{3})$ if $r \pmod 3 = 0$, and is 0 if $r \pmod 3 \neq 0$, where $\phi(x)$ is the Euler totient function.
- We proved that the number of rank r prime non-real elements in the dual canonical basis of $\mathbb{C}[\text{Gr}(4,8)]$ which can be obtained by applying the braid group action to the following two tableaux: $\begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&7\\\hline 6&8\\\hline\end{array}$, $\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline 7&8\\\hline\end{array}$ is $2\phi(\frac{r}{2})$ if $r \pmod 2 = 0$, and is 0 if $r \pmod 2 \neq 0$.

Connection to scattering amplitudes

- In scattering amplitudes in physics, the singularities of planar loop amplitudes in $N = 4$ super Yang-Mills are related to Grassmannian cluster algebras $\mathbb{C}[\mathrm{Gr}(4, n)]$.
- Cluster algebras are related to the branch cut singularities of amplitudes.
- This connection relates the cluster variables of the cluster algebra with the rational symbol letters of the scattering amplitude.
- In the case when the cluster algebra is of infinite type, apart from rational symbol letters, there are algebraic letters.

- In the case of $\mathrm{Gr}(4,8)$, algebraic letters are of the form

$$\frac{z_0 + B_z\sqrt{\Delta}}{z_0 - B_z\sqrt{\Delta}},$$

where $B_z = \frac{2z_1 - z_0 \mathrm{ch}(T)}{\Delta}$, z_0, z_1 are certain cluster variables, $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$

or $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$, $\Delta = A^2 - 4B$, $A = \mathrm{ch}(T)$, $B = \mathrm{ch}(T)^2 - \mathrm{ch}(T \cup T)$,

[Prlina–Spradlin–Stankowicz–Stanojevic–Volovich 2018], [Bourjaily–McLeod–von Hippel–Wilhelm 2018], [Bourjaily–McLeod–Vergu–Volk–Von Hippel–Wilhelm 2019], [Zhang–Li–He 2019], [Drummond–Foster–Gurdogan–Kalousios 2020], [Arkani-Hamed–Lam–Spradlin 2021], [Henke–Papathanasiou 2021].

Connection to scattering amplitudes

- The tableaux $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$ can be obtained by computing rays of facets of Newton polytopes [Arkani-Hamed–Lam–Spradlin 2021], or limit rays via infinite mutation sequences [Drummond–Foster–Gurdogan–Kalousios 2020], [Henke–Papathanasiou 2021], or representation theory of quantum affine algebras [Chang–Duan–Fraser–L. 2020].
- We provide another method [Drummond–Gurdogan–L. 2026] to find the tableaux $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & 7 \\ \hline 6 & 8 \\ \hline \end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}$ by computing stable fixed points of the braid group action.
- Our method also produces stable fixed points for other cluster algebras $\mathbb{C}[\mathrm{Gr}(k, n)]$. We expect that these stable fixed points will have applications in scattering amplitudes in physics.

Thank you!