Cyclically shifted $U_q(\widehat{\mathfrak{sl}}_N)$ algebra and cluster realizations

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4 June, 2026

Quantum groups

- $U_q(\mathfrak{g})$, $U_q(\widehat{\mathfrak{g}})$

  Today: $\mathfrak{sl}_N$

- Symmetry of integrable systems
  R-matrices

- Geometric representation theory
  5d Nekrasov partition functions

Shifted quantum groups

- Depends on weight μ
- $\mathcal{W}$-algebras [BK]
- Quantization of Coulomb branches [BFN]
  Slices in affine Grassmannian [KWWY]
- Integrable systems (Toda system) [FT, FPT]
- Geometric representation theory [N, COZZ]

Cyclically shifted $U_q^{[c]}(\widehat{\mathfrak{sl}}_N)$

- Depends on $c \in \mathbb{Z}/N\mathbb{Z}$
- Geometric representation theory (surface defect) [BJ]
- Today: algebraic approach
- Application to non-shifted algebras

Algebraic tools

- Explicit R-matrices
  Explicit representations
- $U(\mathfrak{g})$: differential operators

$$\mathfrak{sl}_2 : \quad e \mapsto \partial_y, \quad h \mapsto -2y\partial_y + c, \quad f \mapsto -y^2\partial_y + cy$$

- $U_q(\mathfrak{g})$: q-difference operators
  $x_1, \ldots, x_k, \quad T_1, \ldots, T_k, \quad c_1, \ldots, c_l$

$$T_i x_j = q^{\delta_{ij}} x_j T_i, \qquad [x_i, c_j] = [T_i, c_j] = 0$$

- Quantum torus $\qquad X_1, \ldots, X_N, \qquad X_i X_j = q^{2\epsilon_{ij}} X_j X_i$

Global functions

- $\mathfrak{sl}_2$: $e \mapsto \partial_y$, $h \mapsto -2y\partial_y + c$, $f \mapsto -y^2\partial_y + cy$
  $\mathbb{A}^1 \subset \mathbb{P}^1, \quad U(\mathfrak{sl}_2) \to \mathrm{Diff}(\mathbb{P}^1)$

- More generally: $U(\mathfrak{g}) \to \mathrm{Diff}(G/B)$
  Beilinson–Bernstein localization

- $U_q(\mathfrak{g})$: many maps to quantum tori
  Cluster mutations relate tori

$U_q(\widehat{\mathfrak{sl}}_N)$

- Generators: $E_\omega, F_\omega, K_\omega^{\pm 1}, \quad \omega \in \mathbb{Z}/N\mathbb{Z}$
- Relations:

$$K_\omega E_{\omega'} = q^{a_{\omega,\omega'}} E_{\omega'} K_\omega, \quad K_\omega F_{\omega'} = q^{-a_{\omega,\omega'}} F_{\omega'} K_\omega$$

$$[K_\omega, K_{\omega'}] = 0, \quad [E_\omega, F_{\omega'}] = \frac{\delta_{\omega,\omega'}}{q-q^{-1}}(K_\omega - K_\omega^{-1})$$

Serre relations

$a_{\omega,\omega'} = 2\delta_{\omega,\omega'} - \delta_{\omega+1,\omega'} - \delta_{\omega,\omega'+1}$ Cartan matrix

- $U_q(\widehat{\mathfrak{b}})$: generated by $E_\omega, K_\omega^{\pm 1}$

Drinfeld double $\mathfrak{D}_q(\widehat{\mathfrak{b}})$

- $U_q(\widehat{\mathfrak{sl}}_N)$: $E_\omega, F_\omega, K_\omega^{\pm 1}, \quad \omega \in \mathbb{Z}/N\mathbb{Z}$

$$\dots, \quad [E_\omega, F_{\omega'}] = \frac{\delta_{\omega,\omega'}}{q-q^{-1}}(K_\omega - K_\omega^{-1}), \quad \dots$$

- $\mathfrak{D}_q(\widehat{\mathfrak{b}})$: $E_\omega, F_\omega, K_\omega^{\pm 1}, (K'_\omega)^{\pm 1}$

$$\dots, \quad [E_\omega, F_{\omega'}] = \frac{\delta_{\omega,\omega'}}{q-q^{-1}}(K_\omega - K'_\omega), \quad \dots$$

$$K'_\omega E_{\omega'} = q^{-a_{\omega,\omega'}} E_{\omega'} K'_\omega, \quad K'_\omega F_{\omega'} = q^{a_{\omega,\omega'}} F_{\omega'} K'_\omega,$$

- Hopf algebra

$$\Delta(E_\omega) = E_\omega \otimes 1 + K'_\omega \otimes E_\omega, \quad \Delta(F_\omega) = F_\omega \otimes K_\omega + 1 \otimes F_\omega$$

Definition of $U_q^{[c]}(\widehat{\mathfrak{sl}}_N)$

- Generators: $E_\omega, F_\omega, K_\omega^{\pm 1}, (K'_\omega)^{\pm 1}, \quad \omega \in \mathbb{Z}/N\mathbb{Z}$
- Relations:

$$K_\omega E_{\omega'} = q^{a_{\omega,\omega'+c}} E_{\omega'} K_\omega, \quad \dots$$

$$K_\omega K'_{\omega'} = q^{a_{\omega-c,\omega'} - a_{\omega,\omega'}} K'_{\omega'} K_\omega, \quad [K_\omega, K_{\omega'}] = [K'_\omega, K'_{\omega'}] = 0,$$

$$[E_\omega, F_{\omega'}] = \frac{\delta_{\omega+c,\omega'} K_{\omega'} - \delta_{\omega,\omega'} K'_\omega}{q - q^{-1}}.$$

Serre relations

- $\Delta_{c_1,c_2} \colon U_q^{[c]}(\widehat{\mathfrak{sl}}_N) \to U_q^{[c_1]}(\widehat{\mathfrak{sl}}_N) \otimes U_q^{[c_2]}(\widehat{\mathfrak{sl}}_N), \quad c = c_1 + c_2$

Cyclic shift in Drinfeld double

- Group algebra $\mathbb{C}G = \langle g \mid g \in G \rangle$
  Dual Hopf algebra $\mathrm{Fun}[G] = \langle \delta_g \mid g \in G \rangle$
- $G = \mathbb{Z}/N\mathbb{Z}$ acts on $U_q(\widehat{\mathfrak{b}})$
- Drinfeld double

$$\mathfrak{D}_q(\mathbb{C}G \ltimes U_q(\widehat{\mathfrak{b}})) \simeq \mathrm{Fun}[G] \otimes U_q(\widehat{\mathfrak{b}}_-) \otimes \mathbb{C}G \otimes U_q(\widehat{\mathfrak{b}}_+)$$

- $$\bigoplus_{c \in G} U_q^{[c]} = \mathrm{Fun}[G] \otimes U_q(\widehat{\mathfrak{b}}_-) \otimes e \otimes U_q(\widehat{\mathfrak{b}}_+)$$

Cluster seeds s

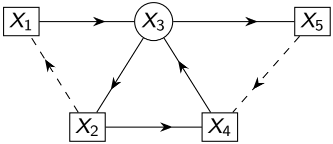
- Quiver: vertices I, frozen nodes $I_{\mathrm{f}} \subset I$

$$\epsilon_{ij} = \#\{i \to j\} - \#\{j \to i\}$$

Dashed arrows: half-edges, incident to frozen nodes

$$X_i X_j = q^{2\epsilon_{ij}} X_j X_i$$

Example

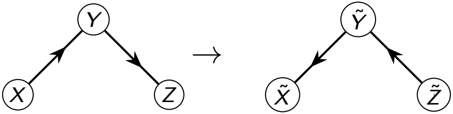


- Laurent algebra $\mathbb{C}_q[\mathcal{X}_{\mathrm{s}}]$ generated by $X_1^{n_1} \cdot \ldots \cdot X_{|I|}^{n_{|I|}}$, $\quad n_i \in \mathbb{Z}$

  Weyl ordered product: $:X_1^{n_1} \cdot \ldots \cdot X_{|I|}^{n_{|I|}}:$

Cluster mutation

- For $k \in I \setminus I_{\mathrm{f}}$: $\mu_k^* \colon \mathbb{C}_q[\mathcal{X}_{\mathrm{s}'}]_{loc} \to \mathbb{C}_q[\mathcal{X}_{\mathrm{s}}]_{loc}$
- Example



$$\tilde{X} = X(1 + q^{-1}Y), \quad \tilde{Y} = Y^{-1}, \quad \tilde{Z} = Z(1 + q^{-1}Y^{-1})^{-1}$$

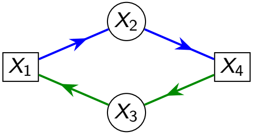
- Quantum global functions $\mathbb{L}_q[\mathcal{X}_{|\mathrm{s}|}]$
  Laurent after any sequence of mutations

Paths

- Path $\mathcal{P} = (Y_0, \ldots, Y_n)$, $\quad Y_0 \to Y_1 \to \cdots \to Y_n$

$$\Sigma[\mathcal{P}] = \sum_{m=0}^{n-1} :Y_0 \cdots Y_m: , \qquad \Psi[\mathcal{P}] = :Y_0 Y_1 \cdots Y_n:$$

- Example: $\mathfrak{sl}_2$

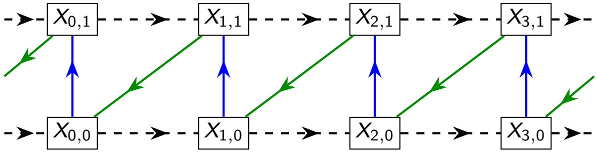


$$(q - q^{-1})E = X_1 + :X_1X_2: , \quad K' = :X_1X_2X_4:$$
$$(q^{-1} - q)F = X_4 + :X_4X_3: , \quad K = :X_4X_3X_1:$$

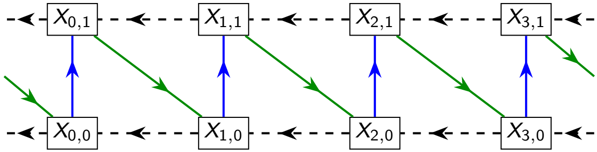
Claim: E, F, K, K' satisfy $\mathfrak{D}_q(\mathfrak{b}_{\mathfrak{sl}_2})$ and belong to $\mathbb{L}_q[\mathcal{X}_{|\mathsf{s}|}]$.

Elementary Representations

$\mathcal{Q}_{4,1}$



$\mathcal{Q}_{4,-1}$



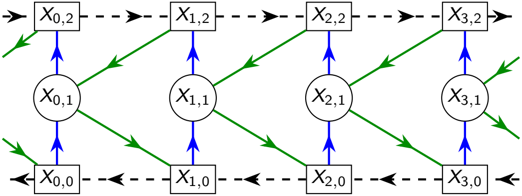
Theorem. For $c = \pm 1$: $\quad \rho^{[c]} \colon U_q^{[c]}(\widehat{\mathfrak{sl}}_N) \to \mathbb{L}_q(\mathcal{X}_{\mathsf{s}_{N,c}})$

$$(q - q^{-1})E_\omega \mapsto X_{\omega,0}, \qquad K'_\omega \mapsto \, :X_{\omega,0}X_{\omega,1}: \,,$$

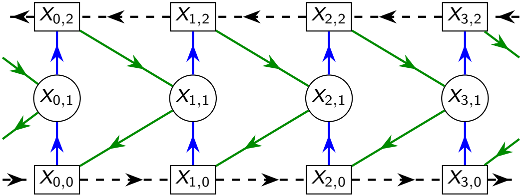
$$(q^{-1} - q)F_\omega \mapsto X_{\omega,1}, \qquad K_\omega \mapsto \, :X_{\omega,1}X_{\omega-c,0}: \,.$$

Parabolic representations

$\mathcal{Q}_{4,-1} * \mathcal{Q}_{4,1}$



$\mathcal{Q}_{4,1} * \mathcal{Q}_{4,-1}$



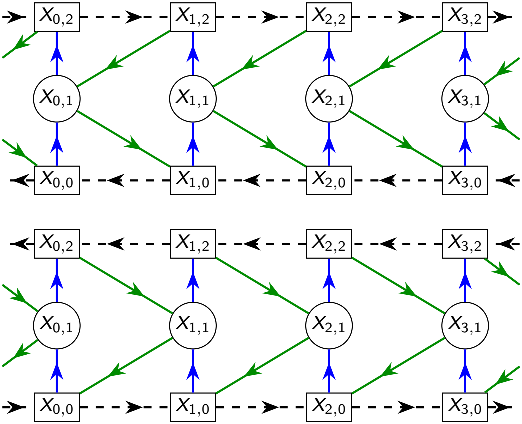
Proposition. $c_1 + c_2 = 0$: $\quad \rho^{[0]} : \mathfrak{D}_q(\widehat{\mathfrak{b}}) \to \mathbb{L}_q(\mathcal{X}_s)$

$$(q - q^{-1})E_\omega \mapsto \Sigma[\mathcal{P}_\omega^E], \quad K'_\omega \mapsto \Psi[\mathcal{P}_\omega^E],$$
$$(q^{-1} - q)F_\omega \mapsto \Sigma[\mathcal{P}_\omega^F], \quad K_\omega \mapsto \Psi[\mathcal{P}_\omega^F].$$

Parabolic representations: intertwiner

$\mathcal{Q}_{4,-1} * \mathcal{Q}_{4,1}$

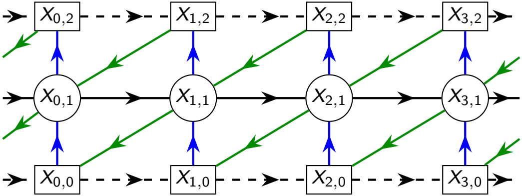
$\mathcal{Q}_{4,1} * \mathcal{Q}_{4,-1}$



- **Proposition.** Intertwiner: $\boldsymbol{\mu}^{1,-1} = \pi_{1,-1} \prod_{\omega \in \mathbb{Z}/N\mathbb{Z}} \mu_{\omega,1}$.
  Formula: product of quantum dilogarithms.
- q-deformed analog of $\mathrm{Diff}(\mathbb{P}^{N-1})$ [I].

$c = 2$ representation

$\mathcal{Q}_{4,1} * \mathcal{Q}_{4,1}$



Proposition. $\rho^{[1+1]}\colon U_q^{[2]}(\widehat{\mathfrak{sl}}_N) \to \mathbb{L}_q(\mathcal{X}_s)$

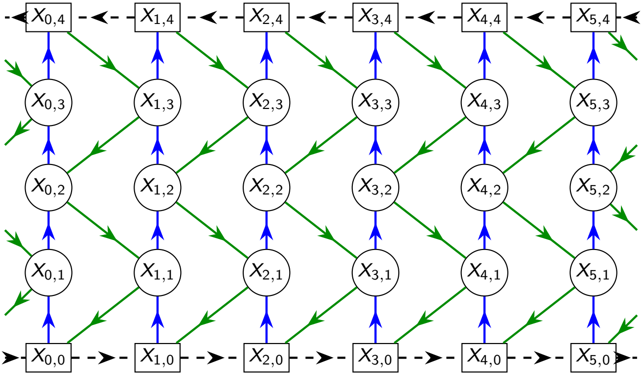
$$(q - q^{-1})E_\omega \mapsto \Sigma[\mathcal{P}_\omega^E], \quad K'_\omega \mapsto \Psi[\mathcal{P}_\omega^E],$$
$$(q^{-1} - q)F_\omega \mapsto \Sigma[\mathcal{P}_\omega^F], \quad K_\omega \mapsto \Psi[\mathcal{P}_\omega^F].$$

Proposition. Intertwiner:

$$\boldsymbol{\mu}^{1,1} = \pi_{1,1}\left(\prod_{\omega=N-1}^{2} \mu_{\omega,1}\right)\mu_{0,1}\mu_{1,1}\left(\prod_{\omega=2}^{N-1}\mu_{\omega,1}\right)$$

Evolution operator

$$\mathcal{Q}_{6,1} * \mathcal{Q}_{6,-1} * \mathcal{Q}_{6,1} * \mathcal{Q}_{6,-1}$$

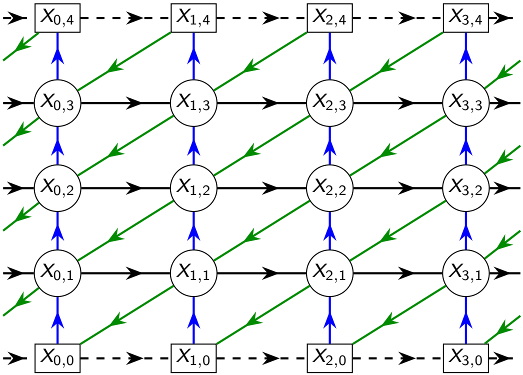


$$R = R^{[1,-1]}_{(2,3)} R^{[-1,-1]}_{(3,4)} R^{[1,1]}_{(1,2)} R^{[-1,1]}_{(2,3)}$$

- Evolution of generalized quantum q-Painlevé system [SO], [H]
- Equation on affine Laumon function (in progress)

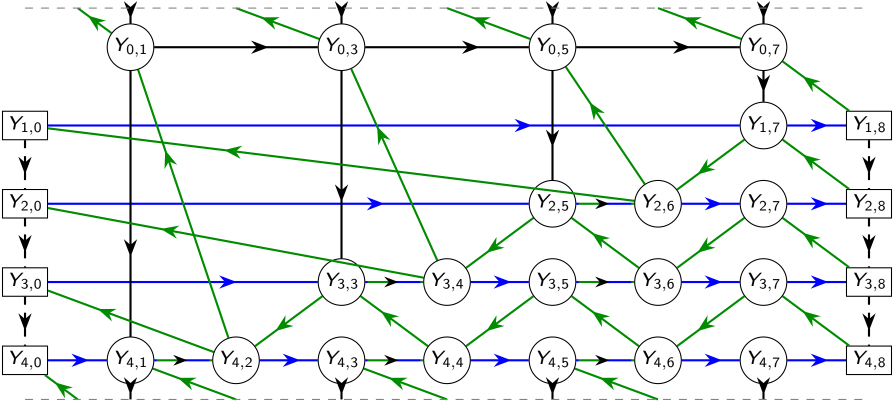
Square representation

$\mathcal{Q}^{*4}_{4,1}$



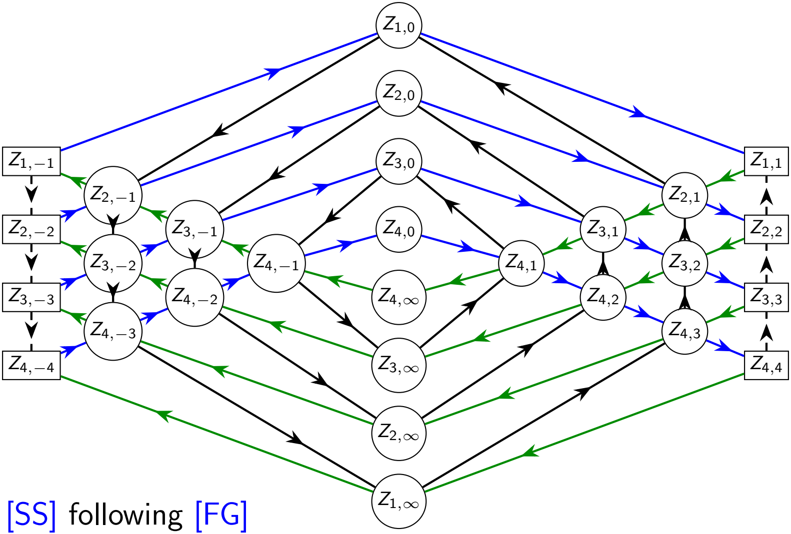
- **Theorem.** $\rho^{[N\cdot 1]}\colon \mathfrak{D}_q(\widehat{\mathfrak{b}}) \to \mathbb{L}_q(\mathcal{X}_{\mathsf{s}^{*N}_{N,1}})$
- Rank $N(N-1) = 2\dim(G/B)$

q-deformed flag manifold representation



- **Theorem.** (a) $\rho^{\mathrm{Ka}}\colon \mathfrak{D}_q(\mathfrak{b}) \hookrightarrow \mathbb{L}_q(\mathcal{X}_{\mathsf{s}_N^{\mathrm{Ka}}})$
- (b) Equivalent to [ANO]
- (c) Equivalent to big representation

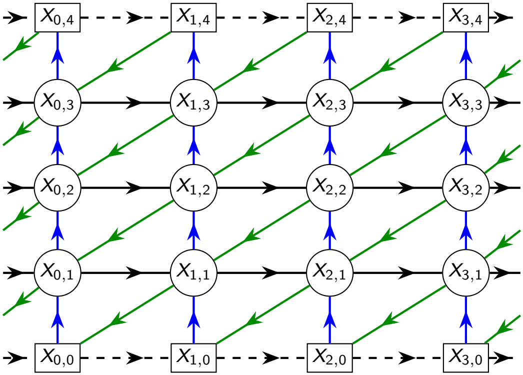
Punctured disk representation



- [SS] following [FG]
- **Theorem.** Equivalent to big representation

R matrix

$\mathcal{Q}^{*4}_{4,1}$



$$\left(\prod_{a=1}^{N}\prod_{b=1}^{N} R^{[1,1]}_{a,b}\right) : \rho^{N\cdot 1}_{\gamma_1,\ldots,\gamma_N} \otimes \rho^{N\cdot 1}_{\delta_1,\ldots,\delta_N} \to \rho^{N\cdot 1}_{\delta_1,\ldots,\delta_N} \otimes \rho^{N\cdot 1}_{\gamma_1,\ldots,\gamma_N}$$

q-analog of [*DM*] generalization of [*DKK*, *VDKK*]

Conclusion

In this talk

- $U_q^{[c]}$ from twisted Drinfeld double
- Cluster realizations from elementary $\rho^{[c]}$
- Equivalences: big / [ANO] / [SS]
- Explicit R-matrices and evolution operators

In progress

- Chiralization
- Proof of conjecture [AHHOSSY]