Drinfeld presentations of Twisted Yangians and their applications

Kang Lu

Based on joint works with Y.-N. Peng, L. Tappeiner, L. Topley, W. Wang, A. Weekes and W. Zhang

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BIMSA
Representation Theory, Integrable Systems and Related Topics

Yangians vs twisted Yangians

Yangian $\mathcal{Y}$

- Lie algebra $\mathfrak{g}$
- R-matrix
- Yang-Baxter equation
- integrable system
- Hopf algebra
- finite W-algebras of type A
- Affine Grassmannian slices
- ...

Twisted Yangian ${}^{\imath}\mathcal{Y}$

- symmetric pair $(\mathfrak{g}, \mathfrak{g}^{\theta})$
- K-matrix
- reflection equation
- boundary integrable system
- coideal subalgebra of the Yangian
- finite W-algebras of classical type
- Affine Grassmannian islices
- ...

Yangians in R-matrix and J presentations

R-matrix presentation [Faddeev-Reshetikhin-Takhtajan]

$\mathcal{Y}_{\mathcal{R}}$ is generated by elements encoded by the generating matrix $T(u)$ subject to

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

and possibly extra central relations.

$$\Delta(T(u)) = T_1(u)T_2(u), \qquad S(T(u)) = T(u)^{-1}$$

J presentation [Drinfeld]

$\mathcal{Y}_{\mathcal{J}}$ is generated by elements x and $J(x)$ for $x \in \mathfrak{g}$ subject to complicated relations.

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \tfrac{1}{2}[x \otimes 1, \Omega_{\mathfrak{g}}].$$

Current algebra $\mathfrak{g}[z]$

- Let e_i, f_i, h_i for $i \in \mathbb{I}$ be Chevalley generators of $\mathfrak{g}$
- Cartan matrix $C = (c_{ij})_{i,j\in\mathbb{I}}$ (focusing on type ADE)
- Set $x^+_{i,r} = e_i z^r$, $x^-_{i,r} = f_i z^r$, $h_{i,r} = h_i z^r$
- Current presentation:

$$
\begin{aligned}
&[h_{i,r}, h_{j,s}] = 0 \quad [x^+_{i,r}, x^-_{j,s}] = \delta_{ij} h_{i,r+s} \\
&[h_{i,0}, x^\pm_{j,s}] = \pm c_{ij} x^\pm_{j,s} \\
&[h_{i,r+1}, x^\pm_{j,s}] - [h_{i,r}, x^\pm_{j,s+1}] = 0 \\
&[x^\pm_{i,r+1}, x^\pm_{j,s}] - [x^\pm_{i,r}, x^\pm_{j,s+1}] = 0 \\
&[x^\pm_{i,r}, x^\pm_{j,s}] = 0 && \text{if } c_{ij} = 0 \\
&\mathrm{Sym}_{r_1,r_2}[x^\pm_{i,r_1}[x^\pm_{i,r_2}, x^\pm_{j,s}]] = 0 && \text{if } c_{ij} = -1
\end{aligned}
$$

Yangian $\mathcal{Y}$ in Drinfeld presentation

- $\{a,b\} := ab + ba$ and $\hbar$ a formal variable

[Drinfeld'87] Drinfeld new/current presentation

$\mathcal{Y}$ is a $\mathbb{C}[\hbar]$-algebra generated by $\xi_{i,r}, x_{i,r}^{\pm}$, $i \in \mathbb{I}$ and $r \in \mathbb{N}$ subject to

$$[\xi_{i,r}, \xi_{j,s}] = 0 \qquad [x_{i,r}^+, x_{j,s}^-] = \delta_{ij}\xi_{i,r+s}$$

$$[\xi_{i,0}, x_{j,s}^{\pm}] = \pm c_{ij} x_{j,s}^{\pm}$$

$$[\xi_{i,r+1}, x_{j,s}^{\pm}] - [\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm\frac{c_{ij}}{2}\hbar\{\xi_{i,r}, x_{j,s}^{\pm}\}$$

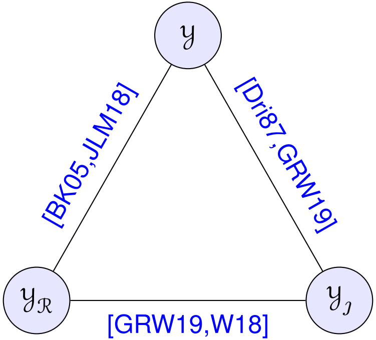
$$[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}] - [x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm\frac{c_{ij}}{2}\hbar\{x_{i,r}^{\pm}, x_{j,s}^{\pm}\}$$

$$[x_{i,r}^{\pm}, x_{j,s}^{\pm}] = 0 \qquad \text{if } c_{ij} = 0$$

$$\mathrm{Sym}_{r_1,r_2}[x_{i,r_1}^{\pm}, [x_{i,r_2}^{\pm}, x_{j,s}^{\pm}]] = 0 \qquad \text{if } c_{ij} = -1$$

- Flat deformation of $\mathcal{U}(\mathfrak{g}[z])$, $\xi_{i,r} \rightsquigarrow h_i z^r$, $x_{i,r}^+ \rightsquigarrow e_i z^r$, $x_{i,r}^- \rightsquigarrow f_i z^r$ as $\hbar \to 0$

Equivalences



Twisted Yangians in R-matrix and J presentations

R-matrix presentation [Sklyanin,Olshanski,Molev-Ragoucy,Guay-Regelskis]

${}^\imath\mathcal{Y}_{\mathcal{R}}$ is generated by elements encoded by the generating matrix $S(u)$ subject to

$$R(u-v)S_1(u)R(u+v)S_2(v) = S_2(v)R(u+v)S_1(u)R(u-v)$$

and some extra relations. ${}^\imath\mathcal{Y}_{\mathcal{R}}$ is a coideal subalgebra of $\mathcal{Y}_{\mathcal{R}}$: e.g. $S(u) = T^t(-u)KT(u)$.

- Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the eigenspace decomposition of an arbitrary involution θ.

J presentation [MacKay02]

${}^\imath\mathcal{Y}_{\jmath}$ is the subalgebra of $\mathcal{Y}_{\jmath}$ generated by all the elements

$$x, \quad B(y) := J(y) - \tfrac{1}{4}[y, C_{\mathfrak{k}}], \quad \text{where } x \in \mathfrak{k},\ y \in \mathfrak{p}.$$

${}^\imath\mathcal{Y}_{\jmath}$ is a right coideal subalgebra: $\Delta(B(y)) = B(y) \otimes 1 + 1 \otimes B(y) - [1 \otimes y, \Omega_{\mathfrak{k}}]$.

- Idea of proving ${}^\imath\mathcal{Y}_{\mathcal{R}} \cong {}^\imath\mathcal{Y}_{\jmath}$ is clear as both are known to the subalgebras of $\mathcal{Y}$

Twisted current algebras $\mathfrak{g}[z]^\theta$

- $\theta : \mathfrak{g} \to \mathfrak{g}, \quad e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \quad h_i \mapsto -h_i$
- Extend θ to $\mathfrak{g}[z]$ by
$$\theta(xz^r) = \theta(x)(-z)^r$$
- The twisted current algebra $\mathfrak{g}[z]^\theta$ is spanned by
$$x_{i,r} := ((-1)^r f_i - e_i)z^r, \qquad t_{i,2r+1} = 2h_i z^{2r+1}$$
- Current presentation:

$$\begin{aligned}
&[t_{i,r}, t_{j,s}] = 0, \qquad t_{i,2r} = 0 \\
&[t_{i,r+1}, x_{j,s}] - [t_{i,r-1}, x_{j,s+2}] = 0 \\
&[x_{i,r+1}, x_{j,s}] - [x_{i,r}, x_{j,s+1}] = -2\delta_{ij}(-1)^r t_{i,r+s+1} \\
&[x_{i,r}, x_{j,s}] = 0 && \text{if } c_{ij} = 0 \\
&\mathrm{Sym}_{k_1,k_2}[x_{i,k_1}, [x_{i,k_2}, x_{j,r}]] = 2(-1)^{k_1} x_{i,k_1+k_2+r} && \text{if } c_{ij} = -1
\end{aligned}$$

iYangian (Twisted Yangian in Drinfeld presentation)

- Recall $\{a, b\} = ab + ba$ and $\hbar$ a formal parameter

Definition [L-Wang-Zhang23-24]

The $\mathbb{C}[\hbar]$-algebra ${}^\imath\mathcal{Y}$ (split type) is generated $h_{i,r}, b_{i,s}$ for $i \in \mathbb{I}, r, s \in \mathbb{N}$ and subject to

$$[h_{i,r}, h_{j,s}] = 0, \qquad h_{i,2r} = 0$$

$$[h_{i,r+1}, b_{j,s}] - [h_{i,r-1}, b_{j,s+2}] = c_{ij}\hbar\{h_{i,r-1}, b_{j,s+1}\} + \frac{c_{ij}^2\hbar^2}{4}[h_{i,r-1}, b_{j,s}]$$

$$[b_{i,r+1}, b_{j,s}] - [b_{i,r}, b_{j,s+1}] = \frac{c_{ij}\hbar}{2}\{b_{i,r}, b_{j,s}\} - 2\delta_{ij}(-1)^r h_{i,r+s+1}$$

$$[b_{i,r}, b_{j,s}] = 0 \qquad \text{if } c_{ij} = 0$$

$$\mathrm{Sym}_{k_1,k_2}\big[b_{i,k_1}, [b_{i,k_2}, b_{j,r}]\big] = (-1)^{k_1+1}[h_{i,k_1+k_2+1}, b_{j,r-1}] \qquad \text{if } c_{ij} = -1$$

Flat deformation of twisted current algebra $\mathcal{U}(\mathfrak{g}[z]^\theta)$, $b_{i,r} \rightsquigarrow x_{i,r},\ h_{i,r} \rightsquigarrow t_{i,r}$ as $\hbar \to 0$

Why is the embedding problem nontrivial?

- The presentation of ${}^\imath\mathcal{Y}$ is obtained
  (1) via Gauss decomposition of $S(u)$ for type A
  (2) via degeneration of affine iquantum groups ${}^\imath\mathcal{U}$ in Drinfeld presentation established by Ming Lu-Wang (ADE) and Zhang (BCFG)
- For type A, ${}^\imath\mathcal{Y}$ is a coideal subalgebra of $\mathcal{Y}$ as ${}^\imath\mathcal{Y} \cong {}^\imath\mathcal{Y}_{\mathcal{R}}$
- For other types, it remains unclear from the degeneration procedure if ${}^\imath\mathcal{Y}$ is a coideal subalgebra of $\mathcal{Y}$
  1. The embedding ${}^\imath\mathcal{U}$ into $\mathcal{U}$ via Drinfeld generators is unknown
  2. Compatibility of filtrations on ${}^\imath\mathcal{U}$ and on $\mathcal{U}$ under the embedding ${}^\imath\mathcal{U} \hookrightarrow \mathcal{U}$ is unclear

Goal

We would like to address this question for other types.

Embedding into Yangian

Theorem [L'25]

Assume that $\mathfrak{g}$ is not of type G_2. ${}^\imath\mathcal{Y}$ is a subalgebra of $\mathcal{Y}$ via the following identification

$$h_{i,1} \mapsto 2\xi_{i,1} - \xi_i^2 + \sum_{\alpha\in\Phi^+} (\alpha, \alpha_i)(x_\alpha^+)^2$$

$$b_{i,0} \mapsto x_i^+ - x_i^-$$

$$b_{i,1} \mapsto x_{i,1}^+ + x_{i,1}^- + \frac{1}{2} \sum_{\alpha\in\Phi^+} \big\{[x_i^+, x_\alpha^+], x_\alpha^+\big\} - \frac{1}{2}\{x_i^+, \xi_i\}$$

Surprisingly, the formulas are similar to the identification of $\mathcal{Y}_\jmath$ with $\mathcal{Y}$ [Drinfeld87]

$$J(\xi_i) \mapsto \xi_{i,1} - \frac{1}{2}\xi_i^2 + \frac{1}{4} \sum_{\alpha\in\Phi^+} (\alpha, \alpha_i)\{x_\alpha^+, x_\alpha^-\}$$

$$J(x_i^\pm) \mapsto x_{i,1}^\pm \pm \frac{1}{4} \sum_{\alpha\in\Phi^+} \big\{[x_i^\pm, x_\alpha^\pm], x_\alpha^\mp\big\} - \frac{1}{4}\{x_i^\pm, \xi_i\}$$

Minimalistic presentation

Theorem [L'25]

If $\mathfrak{g}$ is not of type A_1, $\mathrm{B}_2 \cong \mathrm{C}_2$, or G_2, then ${}^{\imath}\mathcal{Y}$ is generated by $\{h_{i,1}, b_{i,0}, b_{i,1}\}_{i\in\mathbb{I}}$, subject to

$$\begin{aligned}
[h_{i,1}, h_{j,1}] &= 0 \\
[h_{i,1}, b_{j,0}] &= 2c_{ij} b_{j,1} \\
[b_{i,1}, b_{j,0}] - [b_{i,0}, b_{j,1}] &= \tfrac{1}{2} c_{ij} \{b_{i,0}, b_{j,0}\} - 2\delta_{ij} h_{i,1} \\
[b_{i,0}, b_{j,0}] &= 0 && (c_{ij} = 0) \\
\big[b_{i,0}, [b_{i,0}, b_{j,0}]\big] &= -b_{j,0} && (c_{ij} = -1)
\end{aligned}$$

If $\mathfrak{g}$ is of type A_1, $\mathrm{B}_2 \cong \mathrm{C}_2$, or G_2, then an additional relation

$$\Big[h_{i,1}, \big[b_{i,1}, [h_{i,1}, b_{i,1}]\big]\Big] = 4\big[b_{i,1}^2, h_{i,1}\big] \quad \big(\iff [h_{i,1}, h_{i,3}] = 0\big)$$

should be included for any single $i \in \mathbb{I}$.

Coideal structure

Corollary

Assume that $\mathfrak{g}$ is not of type G_2.

1. ${}^\imath\mathcal{Y}$ is a right coideal subalgebra of $\mathcal{Y}$, i.e. $\Delta({}^\imath\mathcal{Y}) \subset {}^\imath\mathcal{Y} \otimes \mathcal{Y}$, and

$$\Delta(h_{i,1}) = h_{i,1} \otimes 1 + 1 \otimes h_{i,1} + 2 \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i)(x_\alpha^+ - x_\alpha^-) \otimes x_\alpha^+.$$

2. The iYangian ${}^\imath\mathcal{Y}$ is isomorphic to the twisted Yangian ${}^\imath\mathcal{Y}_\jmath$ in the J presentation.

Estimation

- $Q_+ := \left\{ \sum_{i\in\mathbb{I}} k_i\alpha_i \;\middle|\; k_i \in \mathbb{N}, \sum_{i\in\mathbb{I}} k_i > 0 \right\}$
- Set $h_i(u) = 1 + \sum_{r\geqslant 0} h_{i,2r+1} u^{-2r-2},\ b_i(u) = \sum_{r\geqslant 0} b_{i,r} u^{-r-1}$

Theorem [L'25]

Let $\mathfrak{g}$ be a simple Lie algebra which is not of type G_2. We have

$$h_i(u) \equiv \xi_i(u)\xi_i(-u) \pmod{\mathcal{Y}_{Q_+}[\![u^{-1}]\!]}$$

$$b_i(u) \equiv \tfrac{1}{2}\{x_i^+(u), \xi_i(-u)\} + x_i^-(-u) \pmod{\mathcal{Y}^{\geqslant 0}_{\alpha_i+Q_+}[\![u^{-1}]\!]}$$

$$\Delta(h_i(u)) \equiv h_i(u) \otimes \xi_i(u)\xi_i(-u) \pmod{{}^\imath\mathcal{Y} \otimes \mathcal{Y}_{Q_+}[\![u^{-1}]\!]}$$

- Affine iquantum groups of split type ABCD and quasi-split type A by Tomasz and Jian-Rong and all quasi-split type by Ming-Lu and Pan
- Application: ${}^\imath q$-character (Tomasz's talk)

Finite W-algebras

- Finite W-algebra $\mathcal{W}(\mathfrak{g}, e)$ depends on $\mathfrak{g}$ and a nilpotent element e in $\mathfrak{g}$
- Whittaker modules
- Quantization of Slodowy slices $e + \mathfrak{g}^f$
- Modular representations of Lie algebras
- Primitive ideals in enveloping algebras
- Zhu algebra of affine $\mathcal{W}$-algebra

Examples

- If $e = 0$, then $\mathcal{W}(\mathfrak{g}, e) \cong \mathcal{U}(\mathfrak{g})$
- If e is principal (regular), then $\mathcal{W}(\mathfrak{g}, e) \cong Z(\mathcal{U}(\mathfrak{g}))$, the center of $\mathcal{U}(\mathfrak{g})$

Finite W-algebras II

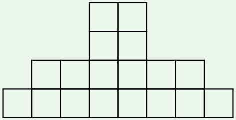
- Generalizing [Ragoucy-Sorba99,Ragoucy01,Brundan-Kleshchev05,Brown09]
- Let $\mathfrak{g}$ be of type BCD and e even
- "even" means the sizes of Jordan blocks of e have the same parity

Theorem [LPTTW25]

For type BC (and D conjecturally), ${}^{\imath}\mathcal{Y}(\sigma)/I_\ell \cong \mathcal{W}(\mathfrak{g}, e)$.

Type C Example

$$\sigma = \begin{pmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 3 \\ 2 & 2 & 0 & 1 \\ 3 & 3 & 1 & 0 \end{pmatrix}, \ell = 8 \qquad \longleftrightarrow$$



- ${}^{\imath}\mathcal{Y}(\sigma) = \langle h_{i,2r+1}, b_{i,s} \rangle_{s \geqslant \sigma_{i,i+1}} \subset {}^{\imath}\mathcal{Y}$

Other applications

- islices (fixed point loci of affine Grassmannian slices) and iCoulomb branches [L-Wang-Weekes'25]
- Quantizations of symmetric quotients of the affine Grassmannian [Bartlett-Przezdziecki-Tappeiner'25]
- Quantized Coulomb branch algebra [Shen-Su-Xiong'25] [Wang'26]
- Equivariant homology of the generalized Steinberg variety of type C [Su-Yang'26]

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Integrable systems?

Thank you!