Rank Recursion for q-Whittaker and Macdonald Operators

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Plan of the talk.

1) Background.

2) Recurrence Relations for q-Whittaker and Macdonald operators

3) Applications to q-TASEP.

Background

In the late 80's, Ian G. Macdonald introduced a new class of symmetric functions

$$P_\lambda(x;q,t) \qquad , x=(x_1,x_2,\dots)$$

λ is an integer partition of N, $\lambda=(\lambda_1,\lambda_2,\dots\lambda_N)$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$

q,t are parameters

$P_\lambda(x;q,t) \in \Lambda_{\mathbb{F}} :=$ ring of symmetric function over $\mathbb{F} = \mathbb{C}(q,t)$

$\uparrow$

Field of rational function over $\mathbb{C}$

(*) P_λ can be uniquely defined from 2 properties:

(1) $$P_\lambda(x;q,t) = m_\lambda + \sum_{\mu < \lambda} R_{\lambda\mu}(q,t)\, P_\mu(x;q,t)$$

"$<$" is the dominance partial ordering on λ ($\mu \leq \lambda \Leftrightarrow \sum \mu_i \leq \sum \lambda_i$)

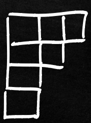
(2) Pairwise orthogonal, i.e. $\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ unless $\lambda = \mu$ w.r.t. the scalar product:

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda(q,t) \quad , \quad z_\lambda(q,t) = \left(\prod_{i \geq 1} i^{m_i} (m_i)!\right) \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$$

power sum symmetric functions

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots \qquad \text{for } \lambda = (1^{m_1} 2^{m_2} \dots)$$

$$p_k := \sum_{i \geq 0} x_i^k$$

 $= (1^2\, 2\, 3)$

Macdonald operators

$$\mathcal{M}_N^r(x_1,\dots,x_N) := t^{\binom{r}{2}} \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\in\{1,\dots,N\}\setminus I}} \frac{tx_i - x_j}{x_i - x_j}\, T_I$$

where $T_I f(x_1,\dots,x_N) = f(x_1,\dots,qx_i,\dots,x_N)$ for $i\in I$

For example,

$$\mathcal{M}_N^1(x_1,\dots,x_N) = \sum_{i=1}^{N} \prod_{j\neq i} \frac{tx_i - x_j}{x_i - x_j}\, T_i$$

Macdonald polynomials are eigenfunctions of Macdonald operators, namely, for $\vec{x} = (x_1, \dots, x_N)$

$$\mathcal{M}_N^r(\vec{x})\, P_\lambda(\vec{x}; q, t) = e_r\left(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \dots, q^{\lambda_N}\right) P_\lambda(\vec{x}; q, t)$$

where $e_r(\vec{x})$ are elementary symmetric polynomials

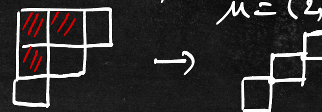
$$e_r(x_1, \dots, x_N) = \sum_{1 \le i_1 < i_2 < \dots < i_r \le N} x_{i_1} x_{i_2} \cdots x_{i_r}$$

At the level of Macdonald polynomials

$$P_\lambda(x_1,\dots,x_N;q,t) = \sum_{\mu \subseteq \lambda} \psi_{\lambda/\mu}(q,t)\, P_\mu(x_1,\dots,x_{N-1};q,t)\, x_N^{|\lambda|-|\mu|}$$

where μ are all partitions s.t. λ/μ are horizontal strips: $\lambda=(3,2,1)$

$\mu=(2,1)$



λ/μ

[Macdonald '98, Chapter VI.7]

[Haglund-Haiman-Loehr '05] for the combinatorial interpretation of $\psi_{\lambda/\mu}(q,t)$

It turns out that there are recurrence relations among Macdonald operators !!!

Recurrence relations for q-Whittaker and Macdonald Operators

Setting $t=0$ and consider the level-1 q-Whittaker operator

$$D_N^1(x_1,\dots,x_N) := \sum_{i=1}^{N} \prod_{j\neq i} \frac{x_i}{x_i-x_j} T_i$$

Then one has the following relation

$$D_N^1(x_1,\dots,x_N) = \frac{x_N}{x_N-x_{N-1}} D_{N-1}^1(x_1,\dots,x_{N-2},x_N) + \frac{x_{N-1}}{x_{N-1}-x_N} D_{N-1}^1(x_1,\dots,x_{N-1})$$

Check: $[T_i]$ for fixed $i=1,\dots,N-2$

$$[T_i]\,\underline{\text{RHS}} = \left(\frac{x_N}{x_N-x_{N-1}}\frac{x_i}{x_i-x_N} + \frac{x_{N-1}}{x_{N-1}-x_N}\frac{x_i}{x_i-x_{N-1}}\right)\prod_{j\neq i}\frac{x_i}{x_i-x_j}$$

$$= \prod_{\substack{j\neq i\\ i\in\{1,\dots,N\}}} \frac{x_i}{x_i-x_j} = [T_i]\,\underline{\text{LHS}}$$

$$D^1_N(x_1,\dots,x_N) = \frac{x_N}{x_N - x_{N-1}} D^1_{N-1}(x_1,\dots,x_{N-2},x_N) + \frac{x_{N-1}}{x_{N-1}-x_N} D^1_{N-1}(x_1,\dots,x_{N-1})$$

Back to the Macdonald operators:

$$\mathcal{M}^1_N(x_1,\dots,x_N) = \frac{t x_{N-1} - x_N}{x_{N-1} - x_N} \mathcal{M}^1_{N-1}(x_1,\dots,x_{N-1})$$

$$+ \frac{t x_N - x_{N-1}}{x_N - x_{N-1}} \mathcal{M}^1_{N-1}(x_1,\dots,x_{N-2},x_N)$$

$$- t\, \mathcal{M}^1_{N-2}(x_1,\dots,x_{N-2})$$

[Di Francesco – V. '26+]: For $r \geq 1$, the q-Whittaker satisfies

$$\boxed{D_N^r(x_1,\dots,x_N) = \sum_{i=N-r}^{N} \prod_{\substack{j=N-r \\ j\neq i}}^{N} \frac{x_j}{x_j - x_i} D_{N-1}^r(x_1,\dots,x_{i-1},x_{i+1},\dots,x_N)}$$

Fix $K \subset \{1,\dots,N\}$, $|K| = r+1$,

$$\boxed{\mathcal{M}_N^r(x_1,\dots,x_N) = \sum_{\substack{J\subseteq K \\ J\neq\phi}} (-1)^{|J|-1} t^{\binom{|J|}{2}} \prod_{\substack{\alpha\in K\setminus J \\ j\in J}} \frac{tx_\alpha - x_j}{x_\alpha - x_j} \mathcal{M}_{N-|J|}^r(\{x_i\}_{i\in\{1,\dots,N\}\setminus J})}$$

$$\mathcal{M}_N^1(x_1,\dots,x_N) = \frac{tx_{N-1}-x_N}{x_{N-1}-x_N}\mathcal{M}_{N-1}^1(x_1,\dots,x_{N-1}) \qquad J=\{N\}$$

$$+\frac{tx_N - x_{N-1}}{x_N - x_{N-1}}\mathcal{M}_{N-1}^1(x_1,\dots,x_{N-2},x_N) \qquad J=\{N-1\}$$

$$-t\,\mathcal{M}_{N-2}^1(x_1,\dots,x_{N-2}) \qquad J=\{N,N-1\}$$

For the q-Whittaker, define $\tilde{D}_N := D_N^1(x_1, \dots, x_{N-1}, x_{N+1})$

[Ahn-Petrov], [Di Francesco-U. '26+]

$$\boxed{(\tilde{D}_N)^k = \sum_{j=0}^{k} \varphi_{q, \frac{x_{N+1}}{x_N}}(j \mid k)\,(D_N)^j\,(D_{N+1})^{k-j}}$$

where $\varphi_{q,\alpha}(j \mid k) := \alpha^j (\alpha; q)_{k-j} \dfrac{(q;q)_k}{(q;q)_j\,(q;q)_{k-j}}$

q-deformed binomial distribution

$$\sum_{j=0}^{k} \varphi_{q,\alpha}(j \mid k) = 1$$

where $(a;q)_k := (1-a)(1-aq)\dots(1-aq^{k-1})$ is the Pochhammer symbol

The Macdonald operators also have a k^{th}-power formula

$$\left(\mathcal{M}_N^r\right)^k = \sum_{\substack{I_1,\dots,I_k \subset [N] \\ |I_\ell| = r}} \left(\prod_{\ell=1}^{k} \prod_{\substack{i \in I_\ell \\ j \in \{1,\dots,N\} \setminus I_\ell}} \frac{t q^{m_{<\ell}(i)} x_i - q^{m_{<\ell}(j)} x_j}{q^{m_{<\ell}(i)} x_i - q^{m_{<\ell}(j)} x_j} \right) T_{I_1} \cdots T_{I_k}$$

where $m_{<\ell}(j) := \sum_{i=1}^{\ell-1} \delta_{j \in I_i}$

Remark: There is a more general notion of q-difference operators [Di Francesco – Kedem '17]

$\mathcal{D}_\alpha(P)$ indexed by $P \in \mathbb{C}(q,t)(x_1,\dots,x_\alpha)^{S_\alpha}$, $\alpha \in \mathbb{Z}_{>0}$

$$\mathcal{D}_\alpha(P) := \frac{1}{\alpha!(N-\alpha)!} \operatorname{Symm}\left(P(x_1,\dots,x_\alpha) \prod_{1\le i\le\alpha<j\le N} \frac{tx_i - x_j}{x_i - x_j} T_1 \cdots T_\alpha \right)$$

where $\operatorname{Symm} f(x_1,\dots,x_n) = \frac{1}{n!} \sum_{\sigma\in S_n} f(x_{\sigma(1)},\dots,x_{\sigma(n)})$

The recurrence relations hold via computation but the trick we use to prove previous result has some techinal issue.

Nevertheless, there's a multi-rank construction via shuffle product [Di Francesco - Kedem '17]. Define

$$\zeta(x) := \frac{1-tx}{1-x} \frac{t-qx}{1-qx}$$

For $P \in \mathbb{C}_{q,t}(x_1,\ldots,x_\alpha)$,
$P' \in \mathbb{C}_{q,t}(x_1,\ldots,x_\beta)$,

$$P * P'(x_1,\ldots,x_{\alpha+\beta}) := \frac{1}{\alpha!\,\beta!} \underset{\text{over } \alpha+\beta}{\mathrm{Symm}} \left(P(x_1,\ldots,x_\alpha) P(x_{\alpha+1},\ldots,x_{\alpha+\beta}) \prod_{1\leq i\leq \alpha < j \leq \alpha+\beta} \zeta(x_i/x_j) \right)$$

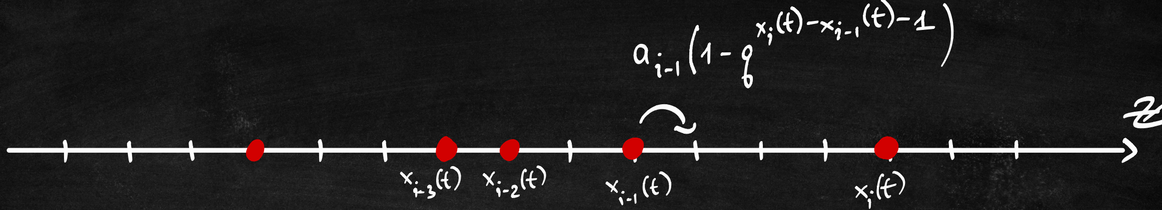
and

$$\mathcal{D}_\alpha(P)\mathcal{D}_\beta(P') = \mathcal{D}_{\alpha+\beta}(P * P')$$

Application to q-deformed Totally Asymmetric Exclusion Processes (q-TASEP)

q-TASEP

[Borodin-Corwin '11]
[Borodin-Corwin-Sasamoto '12], ...



- N-particle system on $\mathbb{Z}$-lattice.

- Each particle has a continuous independent clock, with rate $a_1, \dots, a_N$, parameter $q \in (0,1)$

- An interesting family of observables for q-TASEP is $\{q^{x_i(t)+i}\}_{i\geq 1}$ since they share the same distribution with $\{q^{\lambda_i^{(i)}}\}_{i\geq 1}$ from Macdonald processes.

What's integrable about q-TASEP?

time-scaling to Kardar-Parisi-Zhang equation
[Borodin-Corwin '11]

Duality with the q-Boson Process (using coordinate Bethe Ansatz) with explicit formula for distributions as Fredholm determinant
[Borodin-Corwin '11]
[Borodin-Corwin-Sasamoto '12]

q-TASEP

"Conserved quantities" *
[Petrov '19]
[Petrov-Saenz '22]

Connection to stochastic vertex models where Yang-Baxter equation appeared
[Borodin-Corwin '11]
[Borodin-Corwin-Gorin '14]

This is not a complete list of references

(Ascending) Macdonald Processes

- Measures on triangular arrays of interlacing nonnegative integers (also known as Gelfand-Tsetlin Pattern)

$$\lambda = \{ \lambda_i^{(j)},\ 1 \leq i \leq j \leq N : \lambda_i^{(j)} \leq \lambda_{i-1}^{(j-1)} \leq \lambda_{i-1}^{(j)} \}$$

$\lambda^{(j)} := j^{th}$ level of the triangle and also a partition of length at most j

$$\begin{array}{ccccccccccccccccc}
\lambda_1^{(5)} &&&& \lambda_2^{(5)} &&&& \lambda_3^{(5)} &&&& \lambda_4^{(5)} &&&& \lambda_5^{(5)} \\
& \geq && \geq && \geq && \geq && \geq && \geq && \geq && \geq & \\
&& \lambda_1^{(4)} &&&& \lambda_2^{(4)} &&&& \lambda_3^{(4)} &&&& \lambda_4^{(4)} && \\
&&& \geq && \geq && \geq && \geq && \geq && \geq &&& \\
&&&& \lambda_1^{(3)} &&&& \lambda_2^{(3)} &&&& \lambda_3^{(3)} &&&& \\
&&&&& \geq && \geq && \geq && \geq &&&&& \\
&&&&&& \lambda_1^{(2)} &&&& \lambda_2^{(2)} &&&&&& \\
&&&&&&& \geq && \geq &&&&&&& \\
&&&&&&&& \lambda_1^{(1)} &&&&&&&&
\end{array}$$

Definition of Macdonald measure [Borodin-Corwin '11]

The measure $\mathbb{P}(\lambda)$ depends on $a_1, \dots, a_N \in (0,1)$, $\{\alpha_1, \alpha_2, \dots\}$, $\{\beta_1, \beta_2, \dots\}$, $\gamma > 0$ and 2 parameters $q, t \in (0,1)$

$$\mathbb{P}_{a_1,\dots,a_N;\rho}(\lambda) = \frac{P_{\lambda^{(1)}}(a_1)\, P_{\lambda^{(2)}/\lambda^{(1)}}(a_2) \cdots P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)\, Q_{\lambda^{(N)}}(\rho)}{\Pi(a_1,\dots,a_N;\rho)}$$

single-level marginal

$$\mathbb{P}_{a_1,\dots,a_N;\rho}(\lambda^{(N)}) = \frac{P_{\lambda^{(N)}}(a_1,\dots,a_N)\, Q_{\lambda^{(N)}}(\rho)}{\Pi(a_1,\dots,a_N;\rho)}$$

where $P_{\lambda^{(i)}}$, $P_{\lambda^{(i+1)}/\lambda^{(i)}}$, $Q_{\lambda^{(N)}} := \dfrac{P_{\lambda^{(N)}}}{\langle P_{\lambda^{(N)}}, P_{\lambda^{(N)}}\rangle_{q,t}}$ are Macdonald and skewed Macdonald polynomials

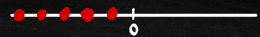
$\Pi(a_1,\dots,a_N;\rho)$ is the normalizing constant $\Pi(a_1,\dots,a_N;\rho) = \sum_{\lambda^{(N)}} P_{\lambda^{(N)}}(a_1,\dots,a_N)\, Q_{\lambda^{(N)}}(\rho)$

ρ is a non-negative specialization depending on $\{\alpha_i\}$, $\{\beta_i\}$ and $\gamma > 0$

Theorem [Borodin-Corwin '11]. Starting from stepped initial data,

$x_i(0) = -i$

$x_0(t) = \infty$



$$\mathbb{E}\left[q^{\lambda_i^{(i)}}\right] = \mathbb{E}\left[q^{x_i(t)+i}\right]$$

observables of Macdonald processes

Fixed time observables of q-TASEP.

Moreover,
$$\left.\frac{(\mathcal{M}_N^1)^y \,\Pi(x_1,\dots,x_n;\rho)}{\Pi(x_1,\dots,x_n;\rho)}\right|_{x_i=a_i} = \mathbb{E}\left[q^{\lambda_N^{(N)} y}\right]$$

single-level moment

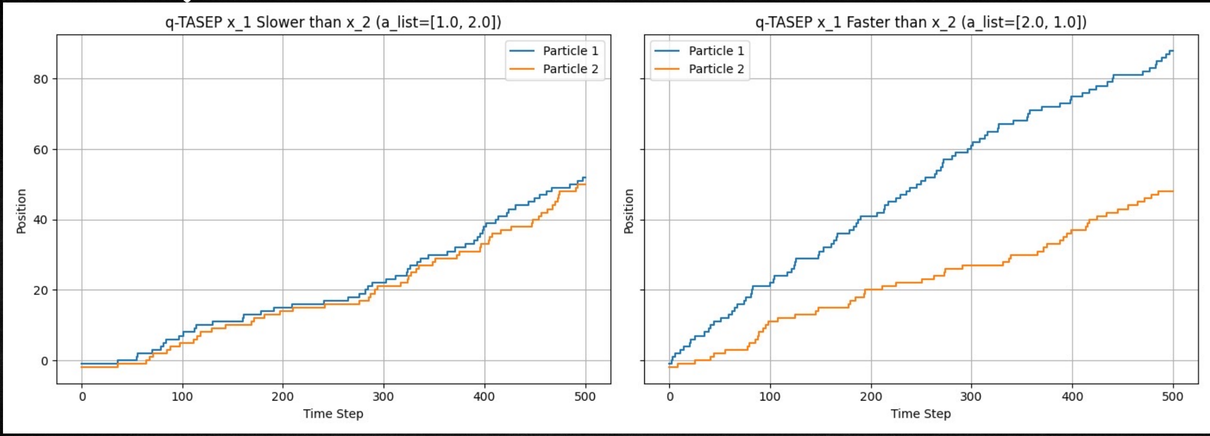
and more generally, [Borodin-Corwin-Gorin-Shakirov '13]

$$\frac{\mathcal{M}_{n_m}^{r_m}\,\mathcal{M}_{n_{m-1}}^{r_{m-1}}\cdots\mathcal{M}_{n_1}^{r_1}\,\Pi(x_1,\dots,x_N;\rho)}{\Pi(x_1,\dots,x_N;\rho)} = \mathbb{E}\left[\prod_{i=1}^{m} e_{r_i}\left(q^{\lambda_1^{(n_i)}}t^{n_i-1}, q^{\lambda_2^{(n_i)}}t^{n_i-2},\dots,q^{\lambda_{n_i}^{(n_i)}}\right)\right]$$

Multi-level moment

Remark: Unlike the situation of fixed rank $\mathcal{M}_N^r$ & $\mathcal{M}_M^r$ does not commute if $N \neq M$.

Some signs of integrability / symmetry from q-TASEP [Petrov '19]



Trajectory of x_2 in both cases are equal in distribution

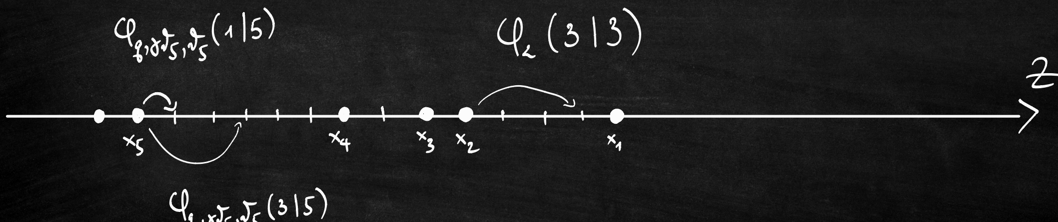
q-Hahn TASEP [Povolotsky '13]

- Discrete Markov process generalizing q-TASEP
- At time t, particle can jump to the right j steps with probability

$$\varphi_{q,\gamma\vartheta_i,\vartheta_i}(j \mid x_i - x_{i-1} - 1), \text{ where } q, \vartheta_i \in (0,1),$$

– q-deformed β binomial distribution

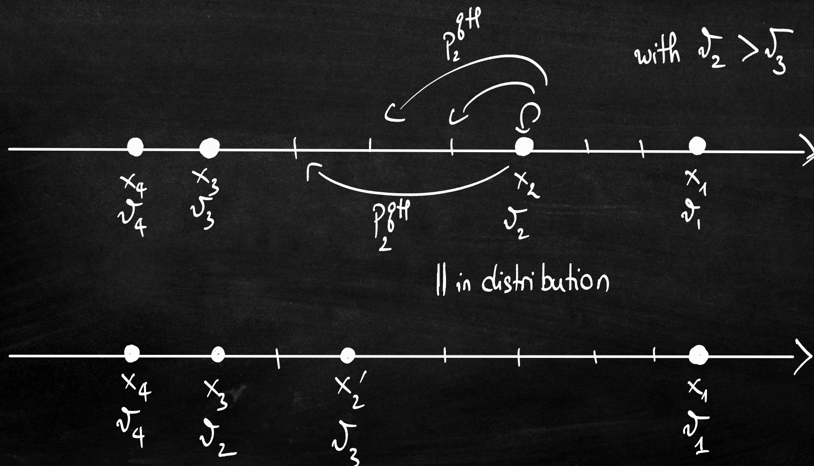
$$\varphi_{q,\mu,\nu}(j \mid m) = \mu^j \frac{(\nu/\mu;q)_j (\mu;q)_{m-j}}{(\nu;q)_m} \frac{(q;q)_m}{(q;q)_{m-j}(q;q)_j}, \quad 0 \le j \le m$$



Theorem [Petrov '19, Petrov-Saenz '22]

The q-Hahn TASEP under the swap $\nu_n \leftrightarrow \nu_{n+1}$, $\nu_n > \nu_{n+1}$, started from stepped initial condition is equivalent to the action of the Markov swapped operator p_n^{qH} on particle x_n, which moves x_n to a random new location x_n' with probability

$$\varphi_{q,\frac{\nu_{n+1}}{\nu_n},\nu_{n+1}}(x_n' - x_{n+1} - 1 \mid x_n - x_{n+1} - 1), \quad x_n' \in \{x_{n+1}+1, \dots, x_n - 1, x_n\}$$

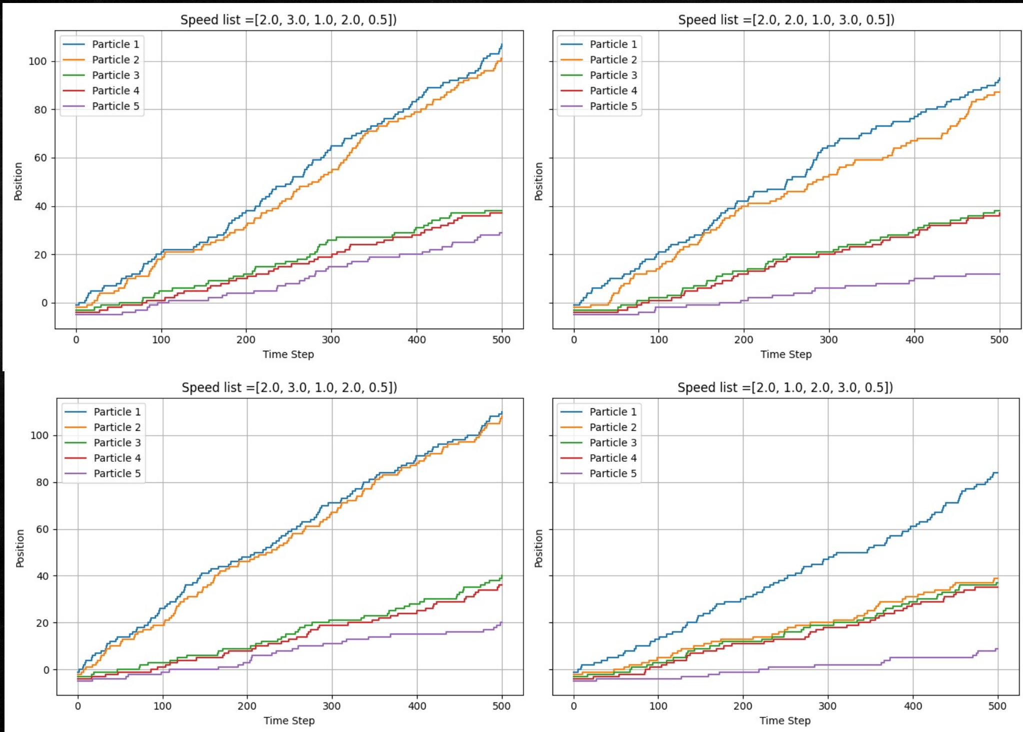


Remark: Markov swapped operator came from a coupling of 2 processes

$\downarrow$ Taking "suitable" limit

q-TASEP

– Can we create Markov operator for coupling with more paritcle?



Thank you so much for your attention!