

Quantum Sasano Systems of Type $D_{2n+2}^{(1)}$

Affine Weyl group symmetry as a machine for special functions

Hajime Nagoya

Kanazawa University

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Three words for the talk: **quantization, symmetry, integral transforms.**

The main message

One-sentence summary

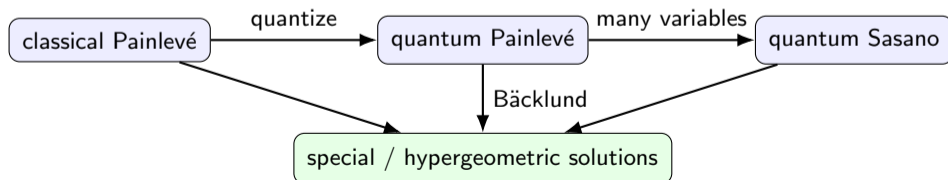
Quantization does not destroy the Painlevé affine Weyl group symmetries. It turns Bäcklund transformations into explicit operators on wavefunctions:

gauge multiplications

and

Euler / fractional integral transforms.

For the Sasano system of type $D_{2n+2}^{(1)}$, this produces a quantum Hamiltonian with D -type symmetry and explicit integral-representation solutions.



Today's content

I. Classical story

Painlevé equations
Hamiltonian form
Affine Weyl group
symmetry

II. Quantum toy model

Quantum P_{II}
Schrödinger equation
Integral transforms

III. Sasano system

$D_{2n+2}^{(1)}$ symmetry
Quantum Hamiltonian
Main theorem

IV. Solutions

Shift operators
Invariant subspaces
Integral representations

The technical formulas are there, but the theme is simple: **symmetry constructs solutions.**

Painlevé equations in one picture

Painlevé's problem

Find new special functions from nonlinear ODEs whose movable singularities are only poles.

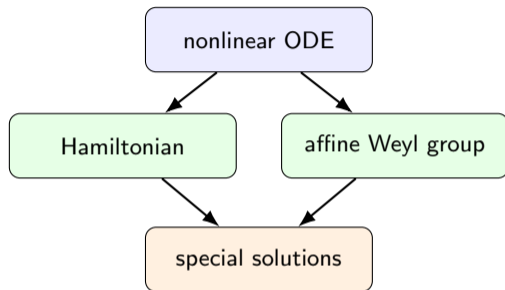
Every Painlevé equation has a Hamiltonian form

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

The hidden organizing principle

Bäcklund transformations form affine Weyl groups and move parameters and solutions together.

Ref: Okamoto (1986).



Airy, Bessel, Gauss, ...

Toy model: P_{II} and $A_1^{(1)}$ symmetry

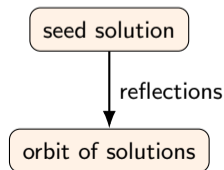
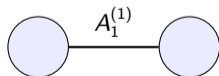
For the second Painlevé equation, one convenient Hamiltonian is

$$H_{II}(q, p, t) = \frac{p^2}{2} - q^2 p - \frac{t}{2} p - \alpha_1 q.$$

A pair of simple reflections acts, for example, by

$$s(q) = q + \frac{\alpha_1}{p}, \quad s(p) = p, \quad s(\alpha_1) = -\alpha_1,$$

$$r(q) = -q, \quad r(p) = -p + 2q^2 + t, \quad r(\alpha_1) = 1 - \alpha_1.$$



Special solutions as symmetry orbits

At special parameter values we often see very classical functions:

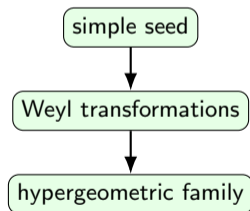
$$P_{II} \left(\frac{1}{2} \right) : \quad q = 0, \quad p = \frac{t}{2},$$

$$P_{II}(0) : \quad q = \frac{u'}{u}, \quad u'' + \frac{t}{2}u = 0 \quad (\text{Airy type}).$$

Ref: Noumi (2004).

Point to remember

The Weyl group does not merely “explain” known special solutions. It is a mechanism for generating whole families of them.



Quantization: the miracle

Canonical quantization replaces variables by non-commuting operators

$$[\hat{p}, \hat{q}] = h, \quad [\hat{q}, t] = [\hat{p}, t] = 0.$$

For example, a quantum Hamiltonian for P_{II} is

$$\hat{H}_{II}(\hat{q}, \hat{p}, t, \alpha_1) = \frac{\hat{p}^2}{2} - \hat{q}\hat{p}\hat{q} - \frac{t}{2}\hat{p} - \alpha_1\hat{q}.$$

What is surprising?

The affine Weyl group action survives the non-commutative deformation, up to scalar shifts.

Not just “ $p \mapsto h\partial_q$ ”, but: **classical symmetries become operators on wavefunctions.**

Ref: Nagoya (2004).

Quantum P_{II} as a Schrödinger equation

Consider

$$\kappa \frac{\partial}{\partial t} \Psi(t, x, \alpha_1) = \hat{H}_{II} \left(x, \frac{\partial}{\partial x}, t, \alpha_1 \right) \Psi(t, x, \alpha_1).$$

Call this equation $\text{Sch}_{II}(\alpha_1)$.

Classical viewpoint

Bäcklund transformations act on (q, p, α) .

Quantum viewpoint

Bäcklund transformations act on wavefunctions Ψ and move the parameter α .

Ref: H.N. (2012).

Two elementary quantum reflections

For quantum P_{II} , the two basic transformations have very concrete forms:

Euler / fractional integral

$$R_s \Psi(x, t) = \int_{\Delta} (x - u)^{\alpha_1 - 1} \Psi(u, t) du.$$

This realizes the inverse derivative hidden in p^{-1} .

Gauge transformation

$$R_r \Psi(x, t) = \exp\left(-\frac{2}{3}x^3 - xt\right) \Psi(-x, t).$$

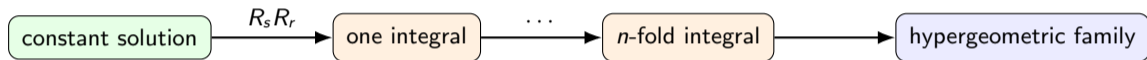
Conjugation by the exponential produces the momentum shift in the reflection r .

If Ψ solves $\text{Sch}_{II}(\alpha)$, then $R_y \Psi$ solves $\text{Sch}_{II}(y(\alpha))$.

Ref: H.N. (2012).

From one seed to many hypergeometric integrals

When $\alpha_1 = -1$, quantum P_{II} has a constant solution. Repeated Bäcklund transformations give nontrivial integral solutions.



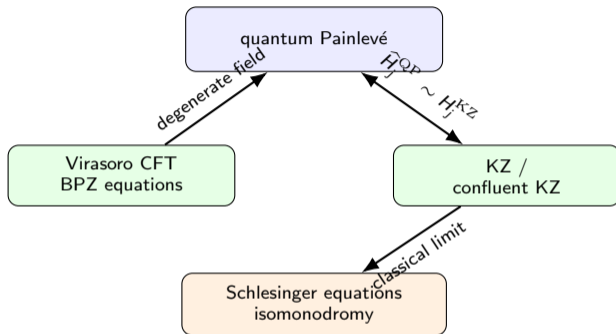
A typical term has the flavor

$$\int_{\Delta} (x - t_n)^{-n\kappa} \prod_{i=1}^{n-1} (t_{i+1} + t_i)^{-i\kappa} \prod_{i=1}^n \exp\left(\frac{2}{3}t_i^3 + t_i x\right) dt_1 \wedge \cdots \wedge dt_n.$$

This is the first appearance of the main pattern: **Weyl symmetry creates integral representations.**

Refs: H.N. (2012).

Where quantum Painlevé appears



Correspondence

- Quantum Painlevé–(confluent) KZ: a Hamiltonian correspondence.

$$\hat{H}_j^{\text{QP}} \sim H_j^{\text{KZ}}.$$

- KZ \rightarrow Schlesinger: a classical-limit statement.

CFT side

Virasoro conformal blocks with a degenerate insertion give BPZ equations and hence quantum Painlevé equations. The relation between Liouville CFT and isomonodromy is deep.

For this talk

Use the CFT and KZ pictures as motivation, but prove an explicit quantum Hamiltonian symmetry statement.

Refs: Reshetikhin (1992); Harnad (1996); Jimbo–H.N.–Sun (2008); H.N. (2011, 2012); H.N.–Yamada (2013).

Why Sasano systems?

Sasano systems of type $D_{2n+2}^{(1)}$ are many-variable generalizations of P_{VI} :

$$H = \sum_{i=1}^n H_i + \sum_{i < j} C_{ij}.$$

Single-particle part

Each H_i resembles a P_{VI} Hamiltonian in (q_i, p_i) .

Coupling part

The terms C_{ij} couple different coordinates and preserve the D -type Weyl symmetry.

They also appear in isomonodromy deformations for Fuchsian systems of spectral type

$$(1^{2n}), \quad (n, n), \quad (n, n), \quad (2n - 1, 1).$$

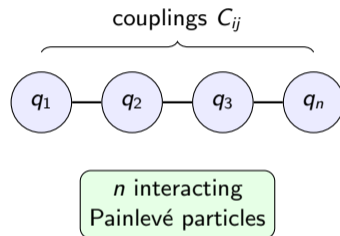
Refs: Sasano (2006); Fuji–Suzuki (2008); Fuji–Inoue–Shinomiya–Suzuki (2013).

Classical Sasano Hamiltonian: structure first

$$H = \sum_{i=1}^n H_i + \sum_{i < j} C_{ij}.$$

$$H_i = q_i(q_i - 1)(q_i - t)p_i^2 - \dots + \alpha_{2i}(\alpha_{2i} + b_0^{(i)})q_i,$$

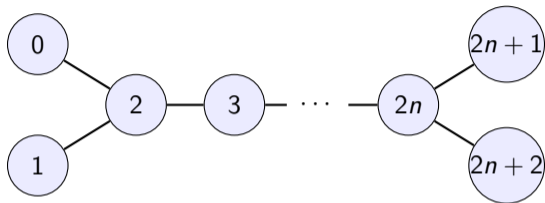
$$C_{ij} = (q_i - t)p_i q_j (\alpha_{2j} + (q_j - 1)p_j).$$



For the audience: first understand the architecture; the exact coefficients can come later.

Refs: Sasano (2006); Fuji–Suzuki (2008).

The affine Dynkin diagram behind the story



Dictionary

Nodes carry parameters $\alpha_0, \dots, \alpha_{2n+2}$. Simple reflections r_k act on both parameters and quantum phase-space operators.

The goal is to realize this whole diagram after quantization.

Ref: Sasano (2006).

Quantizing Sasano: what must be chosen?

Introduce non-commutative operators

$$[\hat{p}_i, \hat{q}_j] = h\delta_{ij}, \quad [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0.$$

1. Single blocks

Quantize each H_i with symmetric ordering.

2. Couplings

Choose ordered coupling terms \hat{C}_{ij} compatible with symmetry.

3. Parameters

Use the $D_{2n+2}^{(1)}$ simple-root parameters.

The resulting quantum Hamiltonian has the form

$$\hat{H} = \sum_{i=1}^n \hat{H}_i + \sum_{i < j} \hat{C}_{ij}, \quad \hat{H}_i = \hat{K}_i + \hat{L}_i + V_i(\hat{q}_i).$$

Algebraic quantum Bäcklund transformations

The simple reflections act by rational transformations.

Even nodes

$$r_{2i}(\hat{q}_i) = \hat{q}_i + \alpha_{2i}\hat{p}_i^{-1}, \quad r_{2i}(\hat{p}_i) = \hat{p}_i.$$

They require an inverse differential operator.

Odd and boundary nodes

$$r_1(\hat{p}_1) = \hat{p}_1 - \alpha_1(\hat{q}_1 - t)^{-1},$$

$$r_{2n+2}(\hat{p}_n) = \hat{p}_n - \alpha_{2n+2}\hat{q}_n^{-1}.$$

They are produced by gauge conjugation.

Parameters transform by $r_k(\alpha_j) = \alpha_j - A_{kj}\alpha_k$.

Main theorem: quantum D -symmetry survives

Theorem

The quantum Hamiltonian \hat{H} of type $D_{2n+2}^{(1)}$ is invariant under the Bäcklund transformations r_k ($k = 0, \dots, 2n + 2$) in the sense

$$r_k(\hat{H}(\alpha, t)) = \hat{H}(\alpha, t) + \delta_{k,1} \frac{\alpha_1 t(1-t)}{\hat{q}_1 - t} + C_k,$$

where C_k is a scalar depending only on parameters and t .

Interpretation

The non-commutative Hamiltonian still carries the full affine Weyl symmetry. The only visible imperfection is an explicit rational anomaly, and it will cancel on the Schrödinger equation.

The anomaly is a feature, not a bug

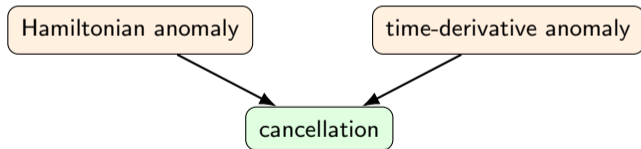
For the quantum Sasano Schrödinger equation

$$ht(1-t)\frac{\partial}{\partial t}\Psi(q,t;\alpha) = \hat{H}\left(q, h\frac{\partial}{\partial q}, \alpha, t\right)\Psi(q,t;\alpha),$$

the term

$$\delta_{k,1}\frac{\alpha_1 t(1-t)}{q_1-t}$$

comes from the time dependence of the gauge factor at the node 1.



This is the clean reason the transformed wavefunction solves the reflected equation.

Bäcklund transformations on wavefunctions

The algebraic transformations are realized analytically by three ingredients.

Gauge factors

$$\tilde{\mathcal{R}}_1 \Psi = (q_1 - t)^{-\alpha_1/h} \Psi$$

$$\tilde{\mathcal{R}}_{2n+2} \Psi = q_n^{-\alpha_{2n+2}/h} \Psi$$

Euler integrals

$$\tilde{\mathcal{R}}_{2i} \Psi = \int_{\Delta_i} (q_i - \tau_i)^{\alpha_{2i}/h-1} \Psi(\dots, \tau_i, \dots) d\tau_i.$$

Phase factors

$$\mathcal{R}_k = e^{\text{scalar phase}} \tilde{\mathcal{R}}_k.$$

They absorb scalar shifts.

Refs: H.N. (2012); this work.

Proof idea in three moves

1. **Conjugate and reflect parameters.**

$$r_k(\widehat{O}) = \pi_k \left(\widetilde{\mathcal{R}}_k \widehat{O} \widetilde{\mathcal{R}}_k^{-1} \right).$$

2. **Apply this to the Hamiltonian.**

$$\widetilde{\mathcal{R}}_k \widehat{H}(\alpha, t) \widetilde{\mathcal{R}}_k^{-1} = \widehat{H}(r_k \alpha, t) + C_k(r_k \alpha, t) - \text{anomaly}.$$

3. **Differentiate $\widetilde{\mathcal{R}}_k \Psi$ in time.** The time derivative produces the same anomaly with the opposite sign; the exponential phase removes C_k .

Ψ solves $\text{Sch}(\alpha) \implies \mathcal{R}_k \Psi$ solves $\text{Sch}(r_k \alpha).$

Shift operators: translations in parameter space

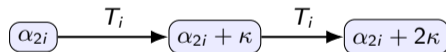
Define

$$T_i = A_i B_i C_i D_i \in W(D_{2n+2}^{(1)}),$$

$$A_i = r_{2i} r_{2i+1} \cdots r_{2n} r_{2n+1} r_{2n+2}, \quad B_i = r_{2n} r_{2n-1} \cdots r_{2i+1}, \quad C_i = r_{2i-1} r_{2i-2} \cdots r_1 r_0, \quad D_i = r_2 r_3 \cdots r_{2i-1},$$

where each factor is a product of simple reflections along the Dynkin diagram. The key effect is

$$T_i(\alpha_{2i}) = \alpha_{2i} + \kappa, \quad T_i(\alpha_j) = \alpha_j \quad (j \neq 2i - 1, 2i, 2i + 1).$$



a discrete flow in the Weyl group

Finite-dimensional invariant subspaces

Set, for $N = (N_1, \dots, N_n) \in \mathbb{Z}_{\geq 0}^n$,

$$\mathcal{V}_N = \text{Span}_{\mathbb{C}(t)} \{q_1^{k_1} \cdots q_n^{k_n} \mid 0 \leq k_m \leq N_m\}.$$

If

$$\alpha_{2i} = -h(N_i + 1) \quad (i = 1, \dots, n),$$

then \mathcal{V}_N is invariant under $\widehat{H}(\alpha, t)$.

Why this matters

This gives finite-dimensional polynomial sectors of the Schrödinger equation: a quantum analogue of special-function solutions.

Integral representations for quantum Sasano

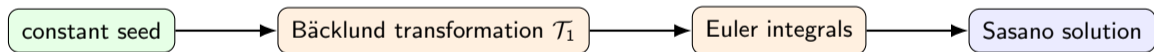
Start from a constant solution at

$$\alpha_{2i} = -h \quad (i = 1, \dots, n).$$

Apply the shift operator

$$\mathcal{T}_1 = \mathcal{R}_2 \mathcal{R}_3 \cdots \mathcal{R}_{2n} \mathcal{R}_{2n+1} \mathcal{R}_{2n+2} \mathcal{R}_{2n} \cdots \mathcal{R}_3 \mathcal{R}_1 \mathcal{R}_0.$$

Then the result is an explicit multiple integral solution of the quantum Sasano equation.



The full formula is long; conceptually it is an Euler-type iterated integral dictated by the Weyl word.

Refs: H.N. (2011); this work.

Why this construction is interesting

1. Non-commutative symmetry

A many-variable quantum Hamiltonian retains affine Weyl symmetry.

2. Explicit operators

The reflections are not abstract: they are gauge and Euler-type integral transformations.

3. Solution factory

Starting from simple seeds, Weyl words generate multiple-integral solutions.

4. Bridges

The picture connects Painlevé, isomonodromy, CFT, and KZ-type equations.

Remark. A quantum isomonodromy equation of type $(1^n), (n-1, 1), (n-1, 1), (1^n)$ is expected to determine the instanton partition function in the presence of the full surface operator in $\mathcal{N} = 2$ $SU(N)$ gauge theory.

Ref: Yamada (2011).

1. Classical Painlevé equations are organized by affine Weyl group symmetries.
2. Quantization preserves these symmetries in a striking analytic form: gauge transformations and Euler integrals.
3. The $D_{2n+2}^{(1)}$ quantum Sasano system realizes this mechanism in many variables and yields explicit integral-representation solutions.

Quantum Painlevé is interesting because symmetry becomes an operator calculus for special functions.

Special functions

Hypergeometric polynomial solutions in the invariant polynomial sectors.

KZ correspondence

KZ equations for the D -type Sasano systems.

Degenerations

A degeneration diagram from quantum Sasano toward lower Painlevé-type systems.

CFT symmetry

Affine Weyl group symmetry of KZ equations.

Thank you!

Questions and comments are welcome.

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Appendix: full quantum Sasano building blocks

$$\widehat{H} = \sum_{i=1}^n \widehat{H}_i + \sum_{i < j} \widehat{C}_{ij}, \quad \widehat{H}_i = \widehat{K}_i + \widehat{L}_i + V_i(\widehat{q}_i).$$

$$\widehat{K}_i = \frac{1}{6} \sum_{\sigma \in \mathfrak{S}_3} f_{\sigma(1)} \widehat{p}_i f_{\sigma(2)} \widehat{p}_i f_{\sigma(3)}, \quad \widehat{L}_i = -\frac{1}{2} \{ \widehat{p}_i, g_i(\widehat{q}_i) \},$$

$$V_i(\widehat{q}_i) = \alpha_{2i} (\alpha_{2i} + b_0^{(i)}) \widehat{q}_i, \quad \widehat{C}_{ij} = 2\widehat{S}_i (\alpha_{2j} \widehat{q}_j + \widehat{T}_j),$$

$$\widehat{S}_i = \frac{1}{2} \{ \widehat{q}_i - t, \widehat{p}_i \}, \quad \widehat{T}_j = \frac{1}{2} \{ \widehat{p}_j, \widehat{q}_j (\widehat{q}_j - 1) \}.$$

Appendix: full gauge transformations

$$\tilde{\mathcal{R}}_0 \Psi = \Psi,$$

$$\tilde{\mathcal{R}}_1 \Psi = (q_1 - t)^{-\alpha_1/h} \Psi,$$

$$\tilde{\mathcal{R}}_{2i+1} \Psi = (q_i - q_{i+1})^{-\alpha_{2i+1}/h} \Psi \quad (1 \leq i \leq n-1),$$

$$\tilde{\mathcal{R}}_{2n+1} \Psi = (q_n - 1)^{-\alpha_{2n+1}/h} \Psi,$$

$$\tilde{\mathcal{R}}_{2n+2} \Psi = q_n^{-\alpha_{2n+2}/h} \Psi,$$

$$\tilde{\mathcal{R}}_{2i} \Psi = \int_{\Delta_i} (q_i - \tau_i)^{\alpha_{2i}/h-1} \Psi(q_1, \dots, \tau_i, \dots, q_n, t; \alpha) d\tau_i.$$

Appendix: full integral representation is long for a reason

$$\begin{aligned} & \int_{\Delta} (q_1 - u_1)^{\kappa/h} (u_1 - t)^{-\alpha_1/h} \prod_{j=2}^n (q_j - u_j)^{(\alpha_{2j+1} \dots \alpha_{2n} + \alpha_{3 \dots 2n+2})/h-1} \\ & \times \prod_{j=1}^{n-1} (u_j - q_{j+1})^{-(\alpha_{2j+2} \dots \alpha_{2n} + \alpha_{3 \dots 2n+2})/h} (u_n - 1)^{-(\alpha_{3 \dots 2n} + \alpha_{2n+1})/h} \\ & \times u_n^{-(\alpha_{3 \dots 2n} + \alpha_{2n+2})/h} \prod_{j=2}^n (u_j - \tau_j)^{\alpha_{3 \dots 2j}/h-1} \prod_{j=2}^n (u_{j-1} - \tau_j)^{\alpha_{3 \dots 2j-1}/h}. \end{aligned}$$

Each exponent records a root-theoretic contribution from a reflection in the Weyl word.