

N-particle ensembles at high temperature via Dunkl and Cherenik operators

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June 2, 2026 @ RTISART: Representation Theory,
Integrable Systems and Related Topics (BIMSA)

Based on joint works with **Florent Benaych-Georges** (CFM),
Vadim Gorin (UC Berkeley) and **Maciej Dolega** (IMPAN).

Outline of the talk

The Gaussian β -ensemble and semicircle distribution

LLN for random β -partitions at high temperature

The limiting measure: moment problem and Jacobi operators

A remark on a deformation of free convolution

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Gaussian Unitary Ensemble

The probability measure on the closed Weyl chamber

$$\mathcal{W}_N := \left\{ (a_1 \geq \dots \geq a_N) \in \mathbb{R}^N \right\}$$

with density

$$\mathbb{P}_N(a_1, \dots, a_N) \propto \prod_{1 \leq i < j \leq N} (a_i - a_j)^2 \prod_{k=1}^N e^{-\frac{1}{2} a_k^2}$$

will be called the **Gaussian Unitary Ensemble (GUE)**.

It is the distribution of eigenvalues of the $N \times N$ complex Hermitian random matrix

$$A_N = \frac{M_N + M_N^*}{2}, \quad M_N = [m_{ij}]_1^N, \quad m_{ij} = \mathcal{N}(0, 1) + \sqrt{-1} \cdot \mathcal{N}(0, 1).$$

Law of Large Numbers for GUE

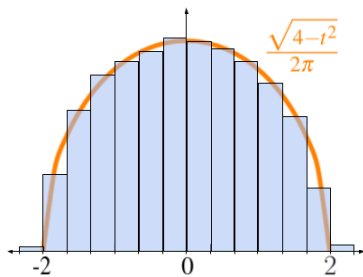
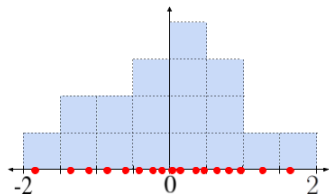
The empirical measures are

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{\frac{a_i}{\sqrt{N}}}, \text{ where } (a_1 \geq \dots \geq a_N) \text{ is } \mathbb{P}_N\text{-distributed.}$$

Theorem (Wigner '55)

The empirical measures μ_N converge weakly, in probability, to the semicircle distribution, with density

$$s(t) := \mathbf{1}_{\{-2 \leq t \leq 2\}} \cdot \frac{\sqrt{4-t^2}}{2\pi}.$$



Moment method and multivariate Bessel functions

The standard proof of Wigner's theorem uses the moment method and matrix representation to find the limits

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}} x^k \mu_N(dx) = \lim_{N \rightarrow \infty} \frac{1}{N^{\frac{k}{2}+1}} \cdot \mathbb{E} \left[\text{Tr}(A_N^k) \right], \text{ for all } k \geq 1.$$

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New ideas [Bufetov–Gorin '15] use the **multivariate Bessel functions**

$$B_{(a_1, \dots, a_N)}(x_1, \dots, x_N) := \prod_{j=1}^{N-1} j! \cdot \frac{\det [e^{a_i x_j}]_{i,j=1}^N}{\prod_{i < j} (x_i - x_j)(a_i - a_j)},$$

s.t. $B_{(a_1, \dots, a_N)}(0^N) = 1$ and are eigenfunctions of differential operators

$$\mathcal{D}_k := \frac{1}{\prod_{i < j} (x_i - x_j)} \circ \sum_{i=1}^N \frac{\partial^k}{\partial x_i^k} \circ \prod_{i < j} (x_i - x_j), \quad k \geq 1.$$

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$$\mathcal{D}_k \left(B_{(a_1, \dots, a_N)}(x_1, \dots, x_N) \right) = \sum_{i=1}^N (a_i)^k \cdot B_{(a_1, \dots, a_N)}(x_1, \dots, x_N)$$

Bessel generating function = kind of Fourier transform

The idea is to associate, $\mathbb{P}_N(a_1, \dots, a_N) \mapsto F_N(x_1, \dots, x_N)$,
to the GUE its **Bessel generating function**:

$$F_N(x_1, \dots, x_N) := \int B_{(a_1, \dots, a_N)}(x_1, \dots, x_N) \mathbb{P}_N(a_1, \dots, a_N) da_1 \cdots da_N.$$

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The moments of empirical measures are exactly the “Taylor coeffs”,
i.e. first apply \mathcal{D}_k , then find the constant term (set $x_i = 0, \forall i$):

$$\mathcal{D}_k F_N \Big|_{x_1 = \dots = x_N = 0} = \mathbb{E}_{\mu_N} \left[\sum_{i=1}^N (a_i)^k \right].$$

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Upshot: The moments of μ_N can be accessed without matrices!

The difficulty now is to study limits of \mathcal{D}_k applied to the BGF
 $F_N(x_1, \dots, x_N)$ (which BTW equals $= e^{(x_1^2 + \dots + x_N^2)/2}$), then
set $x_i = 0, \forall i$, and take the limit, as $N \rightarrow \infty$.

Dunkl operators

A new approach, due to [Benaych-Georges–C.–Gorin '22], uses instead the **Dunkl differential-difference operators**

$$\xi_i := \frac{\partial}{\partial x_i} + \sum_{j: j \neq i} \frac{1}{x_i - x_j} (1 - s_{i,j}),$$

$$\mathcal{P}_k := (\xi_1)^k + \cdots + (\xi_N)^k, \quad k \geq 1.$$

They also satisfy the key identity:

$$\mathcal{P}_k \left(B_{(a_1, \dots, a_N)}(x_1, \dots, x_N) \right) = \sum_{i=1}^N (a_i)^k \cdot B_{(a_1, \dots, a_N)}(x_1, \dots, x_N)$$

The advantage: they admit a 1-parameter “ θ -generalization” ...

Gaussian Beta Ensemble

For any $\theta > 0$, the following distribution on \mathcal{W}_N is the **Gaussian Beta Ensemble** (the random matrix parameter is $\beta = 2\theta$):

$$\mathbb{P}_N^{(\theta)}(a_1, \dots, a_N) \propto \prod_{1 \leq i < j \leq N} (a_i - a_j)^{2\theta} \prod_{k=1}^N e^{-\frac{1}{2}a_k^2}$$

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Motivations:

1. For $\theta = \frac{1}{2}, 2$: $\mathbb{P}_N^{(\theta)}$ = eigenvalue density of Gaussian Orthogonal Ensemble (GOE) and Gaussian Symplectic Ensemble (GSE).
2. $\mathbb{P}_N^{(\theta)}$ is a Boltzmann distribution with logarithmic repulsion and **inverse temperature $\beta = 2\theta$** .
3. Integrable model, e.g. normalization constant \leftarrow Selberg integral.

Bessel generating functions

The relevant multivariate Bessel functions $B_{(a_1, \dots, a_N)}^{(\theta)}(x_1, \dots, x_N)$ are now defined abstractly from the θ -Dunkl operators

$$\xi_i^{(\theta)} := \frac{\partial}{\partial x_i} + \theta \cdot \sum_{j: j \neq i} \frac{1}{x_i - x_j} (1 - s_{ij}),$$

$$\mathcal{P}_k^{(\theta)} := (\xi_1^{(\theta)})^k + \dots + (\xi_N^{(\theta)})^k, \quad k \geq 1,$$

and eigenfunction relations

$$\mathcal{P}_k^{(\theta)} \left(B_{(a_1, \dots, a_N)}^{(\theta)}(x_1, \dots, x_N) \right) = \sum_{i=1}^N (a_i)^k \cdot B_{(a_1, \dots, a_N)}^{(\theta)}(x_1, \dots, x_N)$$

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The relevant Fourier-like transform,

$$F_N^{(\theta)}(x_1, \dots, x_N) := \int B_{(a_1, \dots, a_N)}^{(\theta)}(x_1, \dots, x_N) \mathbb{P}_N^{(\theta)}(a_1, \dots, a_N) da_1 \dots da_N,$$

still satisfies: $\mathcal{P}_k^{(\theta)} F_N^{(\theta)} \Big|_{x_1 = \dots = x_N = 0} = \mathbb{E}_{\mu_N} \left[\sum_{i=1}^N a_i^k \right].$

LLN for $G\beta E$ eigenvalues at fixed temperature

Nothing changes if $\theta > 0$ is fixed: as $N \rightarrow \infty$, then

$$\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{\frac{a_i}{\sqrt{N}}}, \text{ where } (a_1 \geq \dots \geq a_N) \text{ is } \mathbb{P}_N^{(\theta)}\text{-distributed,}$$

converge weakly, in probability, to a **semicircle distribution**.

In the outlier $\theta = 0$ case, the interaction $\prod_{i < j} (a_i - a_j)^{2\theta}$ vanishes and we get **Gaussian distribution** as the limit of empirical measures.

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We were interested in the crossover **high temperature regime**,

$$N \rightarrow \infty, \quad \theta \rightarrow 0^+, \quad N\theta \rightarrow \gamma \in (0, \infty),$$

expecting that: **when** $\gamma \rightarrow \infty$, get semicircle distribution;
when $\gamma \rightarrow 0^+$, get Gaussian distribution.

LLN for $G\beta E$ eigenvalues at high temperature

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

Consider the empirical measures

$$\mu_{N,\theta} := \frac{1}{N} \sum_{i=1}^N \delta_{a_i}, \quad \text{where } (a_1 \geq \dots \geq a_N) \text{ is } \mathbb{P}_N^{(\theta)}\text{-distributed.}$$

In the regime: $N \rightarrow \infty$, $\theta \rightarrow 0^+$, $N\theta \rightarrow \gamma \in (0, \infty)$,
the measures $\mu_{N,\theta}$ converge weakly, in probability, to certain $\mu^{(\gamma)}$.

The density of $\mu^{(\gamma)}$ is known [Allez–Bouchaud–Guionnet '12],

but it is complicated: $\frac{d\mu_\gamma(t)}{dt} = \frac{e^{-t^2/2}}{\sqrt{2\pi} \cdot |\hat{f}_\gamma(t)|^2}$, where

$$\hat{f}_\gamma(t) := \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty f_\gamma(s) e^{its} ds, \quad f_\gamma(s) := \pi \sqrt{\frac{\gamma}{\Gamma(\gamma)}} s^{\gamma-1} \frac{e^{-s^2/2}}{\sqrt{2\pi}}.$$

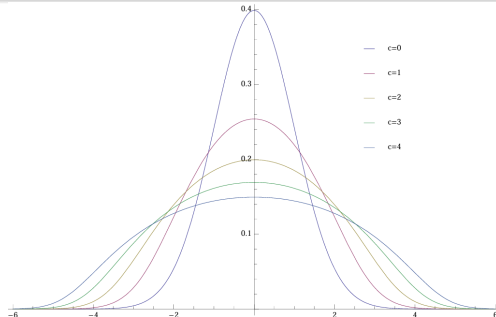
Global asymptotics of $G\beta E$ eigenvalues at high temp

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

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In the limit: $N \rightarrow \infty$, $\theta \rightarrow 0^+$, $N\theta \rightarrow \gamma \in (0, \infty)$,
we have $\mu_{N,\theta} \rightarrow \mu^{(\gamma)}$ weakly, in probability.



Density of $\mu^{(\gamma)}$ for various γ (courtesy of [Allez-Bouchaud-Guionnet '12])

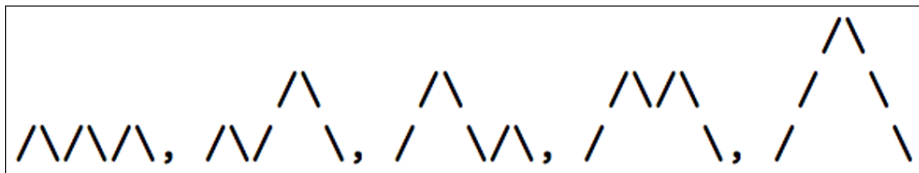
Moments of the limiting measure $\mu^{(\gamma)}$

Theorem (Duy, Shirai '15 & Benaych-Georges, C, Gorin '22)

The limiting measure $\mu^{(\gamma)}$ is uniquely determined by its moments:

$$\int_{-\infty}^{\infty} x^k \mu^{(\gamma)}(dx) = \sum_{\text{Dyck paths } P \text{ of length } k} \text{weight}(P),$$

where: $\text{weight}(P) := \prod_{k \geq 1} (k + \gamma)^{\# \text{down steps from height } k \searrow (k-1)}$.



$$(1+\gamma)^3, (1+\gamma)^2(2+\gamma), (1+\gamma)^2(2+\gamma), (1+\gamma)(2+\gamma)^2, (1+\gamma)(2+\gamma)(3+\gamma)$$

[B-G, C, G '22]: LLN + combinatorial moments \leftarrow control over BGF.

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Discrete Beta Ensembles

We switch gears to discrete **random partitions**.

We are motivated by the **discrete β -ensembles**, due to

[Borodin–Gorin–Guionnet '17], on $\mathcal{W}_{N,\mathbb{Z}} := \{(\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N\}$:

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[Borodin–Gorin–Guionnet '17], on $\mathcal{W}_{N,\mathbb{Z}} := \{(\lambda_1 \geq \dots \geq \lambda_N) \in \mathbb{Z}^N\}$:

- it is convenient to shift the coordinates $\ell_i := \lambda_i - (i - 1)\theta$

so that measures are on N -tuples $(\ell_1 > \dots > \ell_N)$;

- [BGG '17] considered probability distributions of the form

$$\mathbb{P}_N^{(\theta)}(\ell_1 > \dots > \ell_N) \propto \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_i - \ell_j + 1)\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j)\Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^N w_N(\ell_k).$$

The interaction factor comes from Jack symmetric polynomials (characters and spherical functions on symmetric spaces).

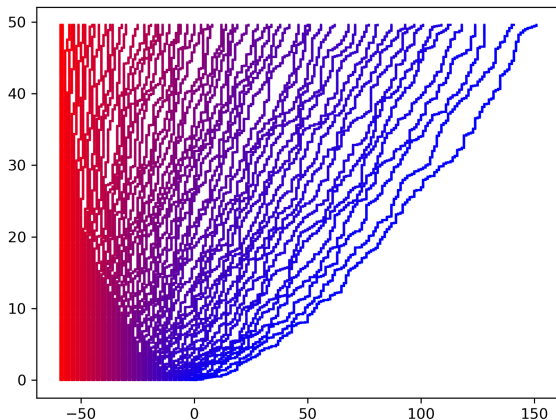
- They proved the LLN for $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i}$ (variational principle).

Discrete β -Dyson Brownian Motion

Another motivation, due to [Gorin–Shkolnikov '15], is a continuous-time, discrete-space, θ -deformed Markov chain

$$(\ell_1(t) > \cdots > \ell_N(t)), \quad t \geq 0,$$

regarded as the **discrete β -Dyson Brownian motion**.



Initial condition: $\lambda(0) = (0, \dots, 0) \Leftrightarrow \ell_i(0) = -(i-1)\theta$.

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regarded as the **discrete β -Dyson Brownian motion**:

- [discrete \rightarrow contin. space limit] gives β -Dyson Brownian motion;
- random evolution of N non-intersecting particles (Doob h -transform of Poisson random walks if only if $\theta = 1$);
- if at time 0, we have $\lambda(0) = (0, \dots, 0) \Leftrightarrow \ell_i(0) = -(i-1)\theta$, then fixed-time distribution at T is the discrete β -ensemble with

$$w_N(\ell) = \frac{T^\ell}{\Gamma(\ell + (N-1)\theta + 1)}, \quad \ell \geq 0.$$

Cherednik operators, Jack polys & generating functions

The **Jack symmetric polys.** $J_{\lambda}^{(\theta)}(x_1, \dots, x_N) = J_{(\lambda_1, \dots, \lambda_N)}^{(\theta)}(x_1, \dots, x_N)$ are defined by

$$\left(\sum_{i=1}^N (\tilde{\xi}_i^{(\theta)})^k \right) J_{\lambda}^{(\theta)}(x_1, \dots, x_N) = \sum_{i=1}^N (\ell_i)^k \cdot J_{\lambda}^{(\theta)}(x_1, \dots, x_N), \quad k \geq 1,$$

where the **Cherednik operators** $\tilde{\xi}_1^{(\theta)}, \dots, \tilde{\xi}_N^{(\theta)}$ are

$$\tilde{\xi}_i^{(\theta)} := \theta(1 - i) + x_i \frac{\partial}{\partial x_i} + \theta \sum_{j=1}^{i-1} \frac{x_i}{x_i - x_j} (1 - s_{i,j}) + \theta \sum_{j=i+1}^N \frac{x_j}{x_i - x_j} (1 - s_{i,j}).$$

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Our results apply to measures with “nice” (analytic near 1^N)

Jack generating functions:

$$F_N^{(\theta)}(x_1, \dots, x_N) := \sum_{\lambda} \mathbb{P}_N^{(\theta)}(\lambda) \frac{J_\lambda^{(\theta)}(x_1, \dots, x_N)}{J_\lambda^{(\theta)}(1^N)}.$$

A variation was used for LLN/CLT at fixed temperature [Huang '20].

For **high temperature**, we proved ...

LLN for random θ -partitions at high temperature

Theorem (C.-Dolega '25; part #1)

Let $\{\mathbb{P}_N\}_{N \geq 1}$ be measures on partitions $(\lambda_1 \geq \dots \geq \lambda_N)$, with empirical measures $\mu_N := \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i}$, where $\ell_i := \lambda_i + \theta(i-1)$.

If the JGF's $\{F_N^{(\theta)}(x_1, \dots, x_N)\}_{N \geq 1}$ satisfy:

- $\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{1}{(\ell-1)!} \frac{\partial^\ell}{\partial x_1^\ell} \ln(F_N^{(\theta)}) \Big|_{x_1=\dots=x_N=1} = \kappa_\ell^{(\gamma)}$, for all $\ell \geq 1$.
- $\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{\partial^r}{\partial x_{i_1} \dots \partial x_{i_r}} \ln(F_N^{(\theta)}) \Big|_{x_1=\dots=x_N=1} = 0$, for all mixed derivs.

Then there is a prob. measure $\mu^{(\gamma)}$ with finite moments m_1, m_2, \dots

s.t. $\lim_{N \rightarrow \infty} \mu_N = \mu^{(\gamma)}$ in the sense of moments, in probability.

LLN for random θ -partitions at high temperature

Theorem (C.-Dolega '25; part #1). LLN for empirical measures if

- $\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{1}{(\ell - 1)!} \frac{\partial^\ell}{\partial x_1^\ell} \ln \left(F_N^{(\theta)} \right) \Big|_{x_1 = \dots = x_N = 1} = \kappa_\ell^{(\gamma)}$, for all $\ell \geq 1$.
- $\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} \ln \left(F_N^{(\theta)} \right) \Big|_{x_1 = \dots = x_N = 1} = 0$, for all mixed derivs.

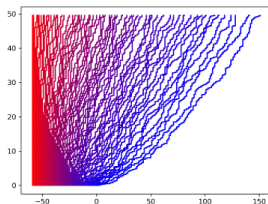
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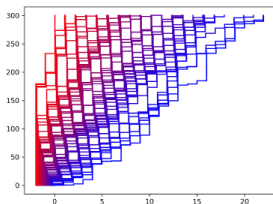
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- $\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{\partial^r}{\partial x_{i_1} \cdots \partial x_{i_r}} \ln \left(F_N^{(\theta)} \right) \Big|_{x_1 = \dots = x_N = 1} = 0$, for all mixed derivs.

Example: For fixed-time T distribution of discrete β -DBM from (0^N) :

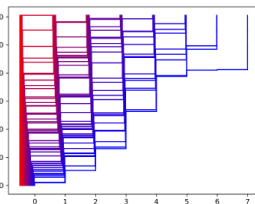
$$F_N^{(\theta)}(x_1, \dots, x_N) = e^{T \cdot \sum_{i=1}^N (x_i - 1)} \Rightarrow \kappa_\ell^{(\gamma)} = \delta_{\ell,1} \cdot T$$



(A) $\theta = 1, N = 60$



(B) $\theta = \frac{2}{3}, N = 60$



(C) $\theta = \frac{1}{2N}, N = 60$

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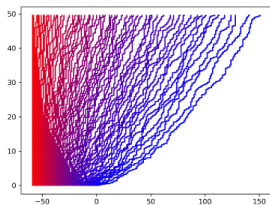
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Fixed-time distribution of the discrete β -DBM

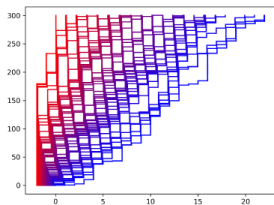
Fixed-time T distribution of discrete β -DBM, started at (0^N) , is:

$$\mathbb{P}_N^{(\theta; T)}(\ell_1 > \dots > \ell_N) \propto \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_i - \ell_j + 1) \Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j) \Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^N \frac{T^{\ell_k}}{\Gamma(\ell_k + (N-1)\theta + 1)}$$

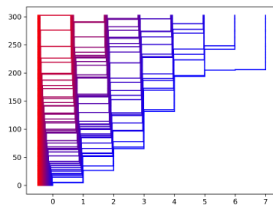
Theorem (C.-Dolega '25; part #1). The empirical measures $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i}$ where $(\ell_1 > \dots > \ell_N)$ is $\mathbb{P}_N^{(\theta; T)}$ -distributed, converge as $N \rightarrow \infty$, $N\theta \rightarrow \gamma$, to some prob. measure $\mu^{(\gamma; T)}$.



(A) $\theta = 1, N = 60$



(B) $\theta = \frac{2}{N}, N = 60$

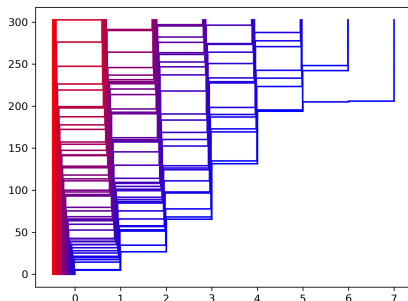


(C) $\theta = \frac{1}{2N}, N = 60$

Law of Large Numbers for random θ -partitions

$$\mathbb{P}_N^{(\theta; T)}(\ell_1 > \dots > \ell_N) \propto \prod_{1 \leq i < j \leq N} \frac{\Gamma(\ell_i - \ell_j + 1) \Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j) \Gamma(\ell_i - \ell_j + 1 - \theta)} \prod_{k=1}^N \frac{T^{\ell_k}}{\Gamma(\ell_k + (N-1)\theta + 1)}$$

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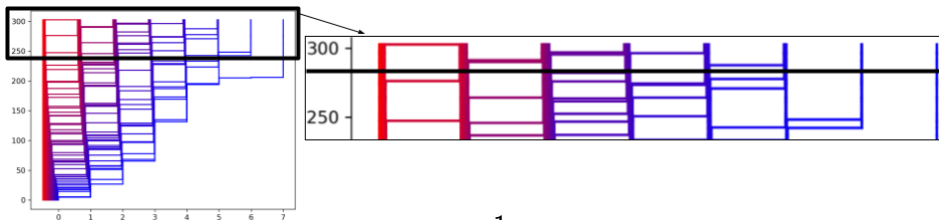


$$N = 60, \quad \theta = \frac{1}{2N} \quad (\gamma = \frac{1}{2}).$$

Law of Large Numbers for random θ -partitions

$$\mathbb{P}_N^{(\theta; T)}(l_1 > \dots > l_N) \propto \prod_{1 \leq i < j \leq N} \frac{\Gamma(l_i - l_j + 1) \Gamma(l_i - l_j + \theta)}{\Gamma(l_i - l_j) \Gamma(l_i - l_j + 1 - \theta)} \prod_{k=1}^N \frac{T^{\ell_k}}{\Gamma(\ell_k + (N-1)\theta + 1)}$$

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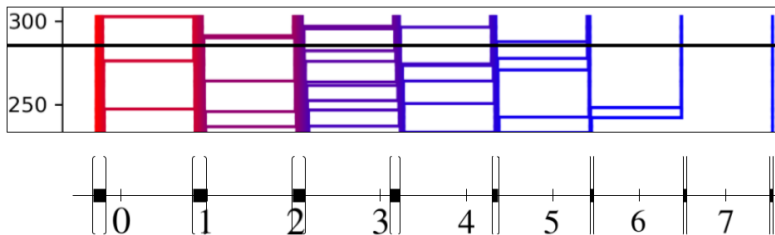


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The support of $\mu^{(\gamma; T)}$

Moments of the limiting measure

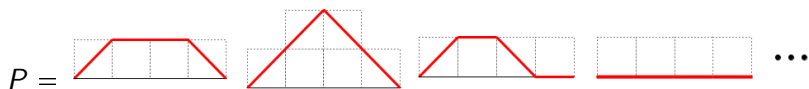
Theorem (C.–Dolega '25; part #1)

The probability measure $\mu^{(\gamma; T)}$ is uniquely determined by its moments, e.g., if $\gamma = T$, then

$$\int_{\mathbb{R}} x^k \mu^{(\gamma; \gamma)}(dx) = \sum_{P \in \text{Motzkin}(k)} \frac{\text{wt}_{\gamma}(P)}{1 + \#\text{horizontal steps at height } 0},$$

where

$$\text{wt}_{\gamma}(P) := \gamma^{\#\text{up steps}} \cdot \prod_{k \geq 0} (k + \gamma)^{\#\text{hor. steps at } k} + \#\text{down steps } k \searrow (k-1)$$



$$\text{wt}_{\gamma}(P) = \gamma(1+\gamma)^3, \quad \gamma^2(2+\gamma)(1+\gamma), \quad \frac{1}{2}\gamma^2(1+\gamma)^2, \quad \frac{1}{5}\gamma^4, \quad \dots$$

What's new?

$$\tilde{\xi}_i = \theta(1-i) + x_i \frac{\partial}{\partial x_i} + \theta \sum_{j=1}^{i-1} \frac{x_j}{x_i - x_j} (1 - s_{i,j}) + \theta \sum_{j=i+1}^N \frac{x_j}{x_i - x_j} (1 - s_{i,j}).$$

Unlike Dunkl operators: Even if $F(\vec{x})$ symmetric, we have

$$(\tilde{\xi}_i)^k F(\vec{x}) \Big|_{\vec{x}=(1^N)} \neq (\tilde{\xi}_j)^k F(\vec{x}) \Big|_{\vec{x}=(1^N)}, \text{ if } i \neq j.$$

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The **Hecke relations**

$$\boxed{\tilde{\xi}_j s_{j-1,j} = s_{j-1,j} \tilde{\xi}_{j-1} - \theta \cdot \text{id}, \quad \forall 2 \leq j \leq N,}$$

reduce the prelimit k -th moment to a polynomial function of

$$\tilde{\xi}_1 F(\vec{x}) \Big|_{\vec{x}=(1^N)}, \quad (\tilde{\xi}_1)^2 F(\vec{x}) \Big|_{\vec{x}=(1^N)}, \quad \dots, \quad (\tilde{\xi}_1)^k F(\vec{x}) \Big|_{\vec{x}=(1^N)}$$

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and $\tilde{\xi}_1 = x_1 \frac{\partial}{\partial x_1} + \theta \sum_{j=2}^N \frac{x_j}{x_1 - x_j} (1 - s_{1,j}) \sim$ Dunkl operator!

Moment generating function of the limiting measure $\mu^{(\gamma; T)}$

The moments $m_k^{(\gamma; T)} := \int_{-\infty}^{\infty} x^k \mu^{(\gamma; T)}(dx)$ can be assembled into:

Theorem (C.-Dolega '25; part #2)

As formal power series in z^{-1} ,

$${}_1F_1(\gamma; z; -T) = \exp \left[\gamma \cdot \tilde{\mathcal{L}} \left\{ \sum_{k=1}^{\infty} \frac{m_k^{(\gamma; T)}}{k!} (-\mathbf{x})^k - \frac{e^{\gamma \mathbf{x}} - \gamma \mathbf{x} - 1}{\gamma \mathbf{x}} \right\} (z) \right]$$

where:

- ${}_1F_1(a; b; x) := \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$, $(t)_n := t(t+1) \cdots (t+n-1)$,

is the *confluent hypergeometric function*;

- $\tilde{\mathcal{L}}$ is the *formal Laplace transform*:

$$\tilde{\mathcal{L}} \left\{ \sum_{n \geq 0} \frac{f_n}{n!} \mathbf{x}^n \right\} (z) := \sum_{n \geq 0} f_n z^{-n-1}.$$

Density of $\mu^{(\gamma; T)}$ via the zeroes of an entire function

By the **Fourier inversion theorem**:

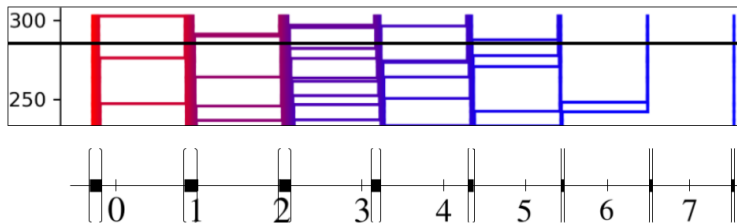
Theorem (C.-Dolega '25; part #2)

For any $\gamma, T > 0$, the density of the limiting measure is

$$\frac{d\mu^{(\gamma; T)}}{dx}(x) = \frac{1}{\gamma} \left\{ \mathbf{1}_{[-\gamma, -\ell_1^{(\gamma; T)}]}(x) + \sum_{k=1}^{\infty} \mathbf{1}_{[1-\ell_k^{(\gamma; T)}, -\ell_{k+1}^{(\gamma; T)}]}(x) \right\},$$

where $\ell_1^{(\gamma; T)} > \ell_2^{(\gamma; T)} > \dots$ are the real zeroes of the entire function

$$G^{(\gamma; T)}(z) := \frac{1}{\Gamma(z)} \cdot {}_1F_1(\gamma; z; -T), \quad z \in \mathbb{C}.$$



Density of $\mu^{(\gamma; T)}$ via the spectrum of a Jacobi operator

$G^{(\gamma; T)}(z)$ is the **characteristic function** of a Jacobi matrix:

Theorem (C.–Dolega '25; part #2)

For any $\gamma, T > 0$,

$$\frac{d\mu^{(\gamma; T)}}{dx}(x) = \frac{1}{\gamma} \left\{ \mathbf{1}_{[-\gamma, -\ell_1^{(\gamma; T)}]}(x) + \sum_{k=1}^{\infty} \mathbf{1}_{[1-\ell_k^{(\gamma; T)}, -\ell_{k+1}^{(\gamma; T)}]}(x) \right\},$$

where $(-\ell_1^{(\gamma; T)} > -\ell_2^{(\gamma; T)} > \dots)$ = spectrum of the Jacobi matrix

$$J^{(\gamma; T)} = \begin{bmatrix} T & \sqrt{T(\gamma+1)} & & & \\ \sqrt{T(\gamma+1)} & T+1 & \sqrt{T(\gamma+2)} & & \vdots \\ & \sqrt{T(\gamma+2)} & T+2 & \sqrt{T(\gamma+3)} & \vdots \\ & & \sqrt{T(\gamma+3)} & T+3 & \vdots \\ & \dots & \dots & & \ddots \end{bmatrix}$$

Outline of the talk

The Gaussian β -ensemble and semicircle distribution

LLN for random β -partitions at high temperature

The limiting measure: moment problem and Jacobi operators

A remark on a deformation of free convolution

Discrete β -DBM with arbitrary initial condition

Assume the discrete β -DBM has initial conditions

$$\ell^{(N)}(0) := \left(\ell_1^{(N)}(0) > \dots > \ell_N^{(N)}(0) \right), \text{ s.t. } \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i^{(N)}(0)} \xrightarrow{N \rightarrow \infty} \nu.$$

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Theorem (C.–Dolega '25; part #2)

Let $\ell^{(N)}(T) = (\ell_1^{(N)}(T) > \dots > \ell_N^{(N)}(T))$ be the Markov chain at time $T > 0$. Then

$$\lim_{\substack{N \rightarrow \infty \\ N\theta \rightarrow \gamma}} \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i^{(N)}(T)} = \mu^{(\gamma; T; \nu)},$$

where $\mu^{(\gamma; T; \nu)}$ is uniquely determined by its moments, equivalently, by the following equalities:

$$\kappa_n^{(\gamma)} \left[\mu^{(\gamma; T; \nu)} \right] = \kappa_n^{(\gamma)} \left[\mu^{(\gamma; T)} \right] + \kappa_n^{(\gamma)} [\nu], \text{ for all } n \geq 1.$$

A deformation of (quantized) free convolution

Question

Let $\gamma > 0$. Given prob. measures μ, ν with $\kappa_n^{(\gamma)}[\mu], \kappa_n^{(\gamma)}[\nu] < \infty$, does there exist a probability measure $\mu \boxplus^{(\gamma)} \nu$ such that

$$\kappa_n^{(\gamma)}[\mu \boxplus^{(\gamma)} \nu] = \kappa_n^{(\gamma)}[\mu] + \kappa_n^{(\gamma)}[\nu], \text{ for all } n \geq 1?$$

Our theorem answers YES, if $\mu = \mu^{(\gamma, T)}$ and ν is of the form $\nu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i^{(N)}(0)}$.

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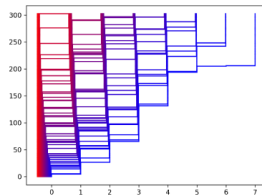
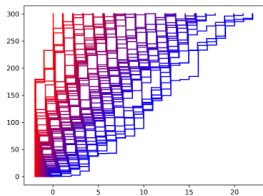
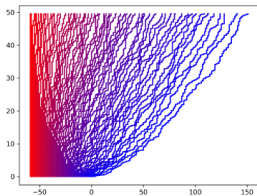
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Conjecture

The answer is always YES.

The question is a limiting case of the conjecture of [Stanley '89] on the positivity of Littlewood-Richardson coeffs. of Jack polynomials.

$\mu \boxplus^{(\gamma)} \nu$ would be a γ -deformed (quantized) free convolution ([Voiculescu '92], [Speicher '94], [Bufetov–Gorin '15]).



Thank you!

