# Integrable geodesic flows on cones over Riemannian manifolds

Andrey E. Mironov and Siyao Yin

Sobolev Institute of Mathematics

Representation Theory, Integrable Systems and Related Topics
June 1–June 5, 2026
BIMSA, Huairou, Beijing, China

## Plan of the Talk

- Geodesic flow
- Examples of integrable geodesic flows
- Obstructions to integrability
- Geodesic flows on the cones over Riemannian manifolds: Superintegrability and complete integrability

## Geodesic Flow on a Closed Riemannian Manifold

- Let $(\Gamma, g)$ be a closed smooth Riemannian manifold, $\dim = n$. Given $(x_0, v_0) \in T\Gamma$, let $\gamma$ be the geodesic with $\gamma(0) = x_0$, $\dot{\gamma}(0) = v_0$. The geodesic flow is defined by

$$\rho^s : T\Gamma \to T\Gamma, \qquad \rho^s(x_0, v_0) = (\gamma(s), \dot{\gamma}(s)).$$

- Equivalently, it is the Hamiltonian flow on $T^*\Gamma$

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i},$$

with

$$H = \frac{1}{2} g^{ij}(x) p_i p_j.$$

## Complete Integrability

- A function $G : T^*\Gamma \to \mathbb{R}$ is a *first integral* if

$$\frac{dG}{dt} = \{H, G\} = \sum_{i=1}^{n} \left( \frac{\partial G}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial x^i} \frac{\partial G}{\partial p_i} \right) = 0.$$

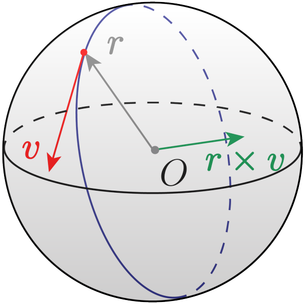
- The geodesic flow is called *completely integrable* if there exist

$$G_1 = H, G_2, \ldots, G_n$$

  functionally independent a.e. on $T^*\Gamma$ and pairwise in Poisson involution:

$$\{G_i, G_j\} = 0.$$

# Geodesic Flow on the Round Sphere $\mathbb{S}^2 \subset \mathbb{R}^3$



$$F = r \times v.$$

## Geodesic Flow on the Torus

**Example 1**

$$H = \frac{p_1^2 + p_2^2}{2f(x)}, \qquad F_1 = p_2.$$

**Example 2**

$$H = \frac{p_x^2 + p_y^2}{2\big(f(x) + g(y)\big)}, \qquad F_2 = \frac{g(y)p_1^2 - f(x)p_2^2}{f(x) + g(y)}.$$

**Kozlov's conjecture:** There are no Riemannian metrics on the torus whose geodesic flow admits an irreducible polynomial first integral $F_n$ of degree $n \geq 3$.

**Theorem** (Bialy, M.)
*If the Hamiltonian system has an integral F which is a homogeneous polynomial of degree $n$, then on the covering plane $\mathbb{R}^2$ there exist the global semi-geodesic coordinates $(t, x)$ such that*

$$ds^2 = g^2(t,x)dt^2 + dx^2, \qquad H = \frac{1}{2}\left(\frac{p_1^2}{g^2} + p_2^2\right)$$

*and F can be written in the form:*

$$F_n = \sum_{k=0}^{n} \frac{a_k(t,x)}{g^{n-k}} p_1^{n-k} p_2^k.$$

*Here the last two coefficients can be normalized by the following way:*

$$a_{n-1} = g,\ a_n = 1.$$

*The condition* $\{F, H\} = 0$ *is equivalent to the quasi-linear PDEs*

$$(*) \qquad U_t + A(U)U_x = 0,$$

*where* $U^T = (a_0, \dots, a_{n-1}),\ a_{n-1} = g,$

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & a_1 \\ a_{n-1} & 0 & \dots & 0 & 0 & 2a_2 - na_0 \\ 0 & a_{n-1} & \dots & 0 & 0 & 3a_3 - (n-1)a_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{n-1} & 0 & (n-1)a_{n-1} - 3a_{n-3} \\ 0 & 0 & \dots & 0 & a_{n-1} & na_n - 2a_{n-2} \end{pmatrix}.$$

## Semi-Hamiltonian systems

**Theorem** (Bialy, M.)
*$(*)$ is semi-Hamiltonian system. Namely, there is a regular change of variables*

$$U \mapsto (G_1(U), \ldots, G_n(U))$$

*such that for some $F_1(U), \ldots, F_n(U)$ the following conservation laws hold:*

$$(G_i(U))_x + (F_i(U))_y = 0, \qquad i = 1, \ldots, n.$$

*Moreover, in the hyperbolic domain, where eigenvalues $\lambda_1, \ldots, \lambda_n$ of A(U) are real and pairwise distinct, there exists a change of variables*

$$U \mapsto (r_1(U), \ldots, r_n(U))$$

*such that the system can be written in Riemannian invariants:*

$$(r_i)_x + \lambda_i(r)(r_i)_y = 0, \qquad i = 1, \ldots, n.$$

## Geodesic flow on the 2-torus in elliptic region

**Theorem** (Bialy, M.)
*Let* $n = 4$, *then in the elliptic regions the following alternative holds: either metric is flat or* $F_4$ *is reducible, that is it can be expressed as:*

$$F_4 = k_1 F_2^2 + 2k_2 H F_2 + 4k_3 H^2$$

*where* $F_2$ *is a polynomial of degree 2 which is an integral of the geodesic flow in the elliptic region and* $k_i$ *are constants.*

## Geodesic flow on the 2-torus in elliptic region

The following theorem is crucial in proof of the previous one.

**Theorem** (Bialy, M.)
*Assume that* $\Omega_e = \mathbb{T}^2$ *and assume that for all* $(t, x)$ *the polynomial* $G_4$ *has 4 distinct roots, 2 – complex conjugate* $s_{1,2} = \alpha \pm i\beta$ *and 2 real* $s_{3,4}$*. Assume also that the imaginary part of Riemann invariants* $r_{1,2}$ *does not vanish. Then the real eigenvalues* $\lambda_{3,4} = g s_{3,4}$ *are necessarily genuinely non-linear and therefore the corresponding Riemann invariants are constants. In particular all* $a_i$ *must be constant, and so the metric is flat.*

## Magnetic Geodesic Flow on the 2-torus

$$H = \frac{1}{2} g^{ij}(x) p_i p_j, \qquad \Omega = \Omega(x)\, dx^1 \wedge dx^2.$$

$$\dot{x}^j = \{x^j, H\}_{mg}, \qquad \dot{p}_j = \{p_j, H\}_{mg}, \qquad j = 1, 2$$

**Magnetic Poisson bracket:**

$$\{F, H\}_{mg} = \sum_{i=1}^{2} \left( \frac{\partial F}{\partial x^i} \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial H}{\partial x^i} \right) + \Omega(x) \left( \frac{\partial F}{\partial p_1} \frac{\partial H}{\partial p_2} - \frac{\partial F}{\partial p_2} \frac{\partial H}{\partial p_1} \right).$$

A function $F(x^1, x^2, p_1, p_2)$ is a *first integral* of the magnetic flow if

$$\{F, H\}_{mg} = 0.$$

## Magnetic Geodesic Flows on the 2-torus

**Example:** Let

$$ds^2 = \Lambda(y)(dx^2 + dy^2), \qquad \Omega = -u'(y)dx \wedge dy.$$

Then the magnetic geodesic flow is integrable and the first integral is linear in momenta:

$$F_1 = p_1 + u(y).$$

**Theorem** (Agapov, Bialy, M.)
*There exist real analytic Riemannian metrics on the 2-torus which are arbitrary close to the Liouville metrics (and different from them) and a non-zero analytic magnetic fields such that magnetic geodesic flows on the energy level* $\{H = \frac{1}{2}\}$ *have polynomial in momenta first integral of degree two.*

## Semi-Hamiltonian system

We shall search the integral $F$ in the form

$$F(x,y,\varphi) = \sum_{k=-N}^{k=N} a_k(x,y)e^{ik\varphi}.$$

Let $N=2$. $\{F,H\}_{mg} = 0$ on the fixed energy level $H = \frac{1}{2}$ gives the quasilinear PDEs on $a_j$ and $\Lambda$ of the form

$$(**) \qquad A(U)U_x + B(U)U_y = 0,$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ f & 0 & \Lambda & 0 \\ 2 & 1 & 0 & \frac{g}{2} \\ 0 & 0 & 0 & -\frac{f}{2} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -g & 0 & 0 & -\Lambda \\ 0 & 0 & -\frac{g}{2} & 0 \\ 2 & -1 & \frac{f}{2} & 0 \end{pmatrix},$$

$$U = (\Lambda, u_0, f, g)^T.$$

Magnetic field takes the form: $\Omega = \frac{1}{4}(g_x - f_y).$

# Crucial construction

One can check that

$$U_0(x,y) = \begin{pmatrix} \Lambda_1(x) + \Lambda_2(y) \\ 2\Lambda_2(y) - 2\Lambda_1(x) \\ 0 \\ 0 \end{pmatrix}$$

is the solution of the system $(**)$, where $\Lambda_1(x)$ and $\Lambda_2(y)$ are periodic positive functions: $\Lambda_1(x+1) = \Lambda_1(x),\ \Lambda_2(y+1) = \Lambda_2(y)$. This solution corresponds to the case of geodesic flow of the Liouville metric with zero magnetic field having the first integral of the second degree of the form

$$F_2 = \frac{\Lambda_2(y)p_1^2 - \Lambda_1(x)p_2^2}{\Lambda_1(x) + \Lambda_2(y)}.$$

$\Lambda_1$ and $\Lambda_2$ are assumed to be real analytic functions.

Introduce the following evolution equations:

$$(***) \qquad U_t = A_1(U)U_x + B_1(U)U_y,$$

where

$$A_1 = \begin{pmatrix} g & 0 & 0 & \Lambda \\ -2g & g & 0 & -2\Lambda \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} f & 0 & \Lambda & 0 \\ 2f & f & 2\Lambda & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

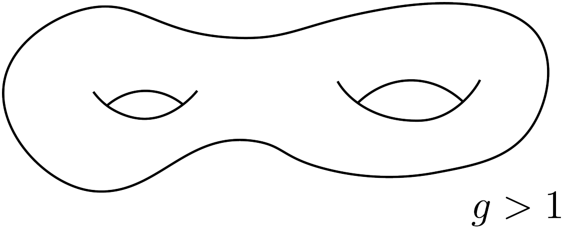
The function $U(x, y, t)$ satisfying

$$U(x, y, t) \mid_{t=0} = U_0(x, y)$$

$$(U(x+1, y, t) = U(x, y+1, t) = U(x, y, t))$$

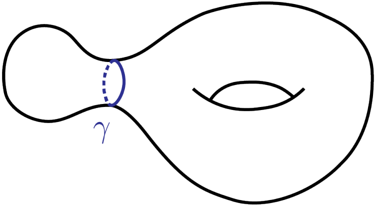
is a solution of $(***)$.

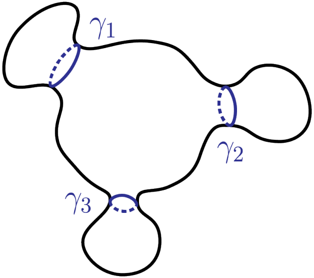
## V. V. Kozlov
## Topological Obstructions to Integrability



V. V. Kozlov, Topological obstructions to the integrability of natural mechanical systems, Sov. Math. Dokl., 20 (1979), 1413–1415.

## S. V. Bolotin
## Geometrical Obstructions to Integrability

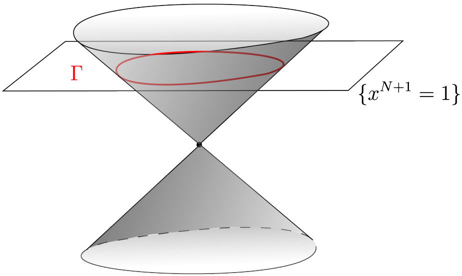




## Obstructions to Integrability on Many-dimensional Real-analytic Manifolds

- **Taimanov (1988)**: Complete integrability in the real-analytic category implies topological restrictions on compact real-analytic manifolds.
- **Butler (1999)**: There exist real-analytic metrics on certain compact nilmanifolds whose geodesic flows have no real-analytic first integrals, but remain integrable in the smooth category.
- **Bolsinov–Taimanov (2000)**: Example of a real-analytic metric on a 3-manifold whose geodesic flow is smoothly integrable, yet has positive topological entropy.

# Cones over Arbitrary Closed $C^3$ Riemannian Manifolds



- Let $\Gamma$ be an arbitrary closed $C^3$ Riemannian manifold, $\dim\Gamma = n$, embedded in $\mathcal{P} = \{x^{N+1} = 1\} \subset \mathbb{R}^{N+1}$ for some $N > n$.

- Let

  $$K = \{\, tp \mid p \in \Gamma,\ t \in \mathbb{R} \,\} \subset \mathbb{R}^{N+1},$$

  be the cone over $\Gamma$.

## Billiards inside Cones of Codimension One

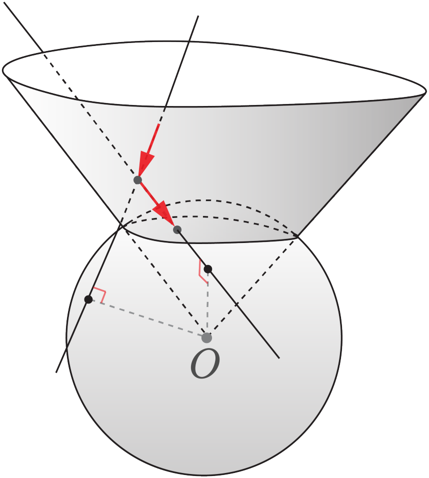
**Theorem** (M., Yin)

1) *The Birkhoff billiard inside $K \subset \mathbb{R}^{N+1}$ admits the first integral*

$$I_B = \|x\|^2 - \frac{\langle x, v\rangle^2}{\|v\|^2},$$

*where $x = (x^1, \dots, x^{N+1})$ denotes the position of the particle and $v = (v^1, \dots, v^{N+1})$ its velocity.*

2) *The spheres centered at the vertex $O \in \mathbb{R}^{N+1}$ of $K$ are caustics of the billiard inside $K$, i.e. lines containing the segments of the trajectory are tangent to the same sphere.*


$O$

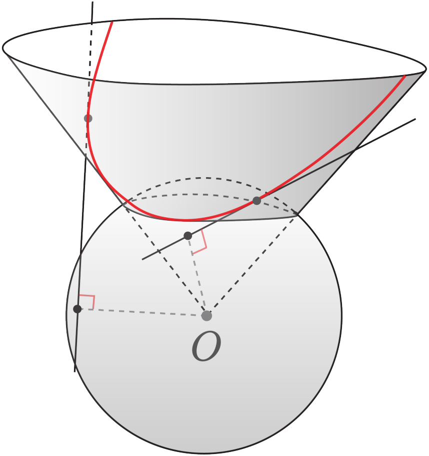
## Geodesic flows on cones

### Theorem 1

Let $\gamma(s)$ be a geodesic on a cone $K \subset \mathbb{R}^{N+1}$. Then the quantity

$$I = \|\gamma(s)\|^2 - \frac{\langle \gamma(s), \gamma'(s) \rangle^2}{\|\gamma'(s)\|^2}$$

remains constant along $\gamma$. Geometrically, $I$ represents the squared distance from the vertex $O$ to the tangent line of $\gamma$.


$O$

## Geodesics on Cones: Two Classes

**Lemma 1**

Every geodesic $\gamma(s)$ on $K$ with $\|\gamma'(s)\| = 1$, after an appropriate shift of parameter, belongs to one of the following two classes:

(i) **Radial geodesics (generatrices):**

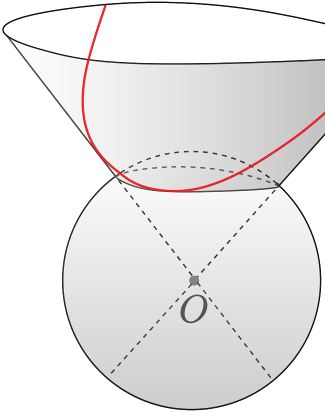
$$\gamma(s) = sp, \quad p \in \Sigma, \quad s \in (-\infty, +\infty),$$

which are straight lines passing through the vertex $O$. They satisfy $I = 0$.

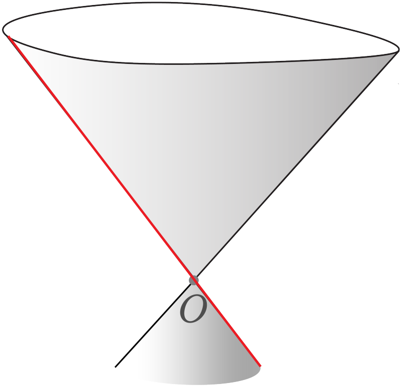
(ii) **Non-radial geodesics:**

$$\gamma(s) = q(s)\sqrt{s^2 + I}, \quad s \in (-\infty, +\infty),$$

where $q(s)$ is some curve on $\Sigma$. They satisfy $I > 0$.



$I > 0$



$I = 0$

## Geodesics on Cones: Projection and Asymptotics

Let

$$\Sigma = K \cap \mathbb{S}^N.$$

Let $\gamma(s) \subset K$ be a non-radial geodesic, with $I = 1$ and $\|\gamma'(s)\| = 1$.

### Lemma 2

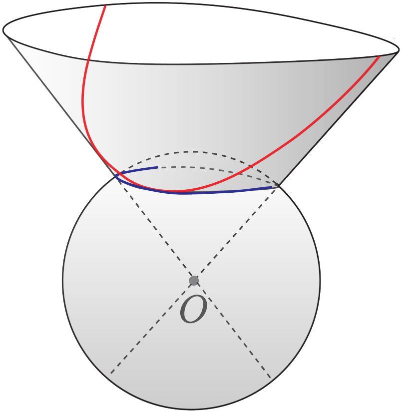
1) The geodesic $\gamma(s)$ is tangent to $\mathbb{S}^N$ at a unique point $\gamma(s_0)$. Its radial projection

$$\tilde{\gamma}(\tilde{s}) = \frac{\gamma(s)}{\|\gamma(s)\|},$$

where

$$s = \tan \tilde{s}, \qquad \tilde{s} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

is a geodesic in $\Sigma$, and $\tilde{s}$ is the arc-length parameter of $\tilde{\gamma}$, $\|\tilde{\gamma}'(\tilde{s})\| = 1$.


O

## Lemma 2 (continued)

2) Conversely, let $\tilde{\gamma}(\tilde{s})$, $\tilde{s} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, be an interval of a geodesic on $\Sigma$ of length $\pi$, with $\tilde{s}$ as its arc-length parameter. Then

$$\gamma(s) = \tilde{\gamma}(\tilde{s})\sqrt{s^2+1}, \qquad \tilde{s} = \arctan s, \qquad s \in (-\infty, +\infty),$$

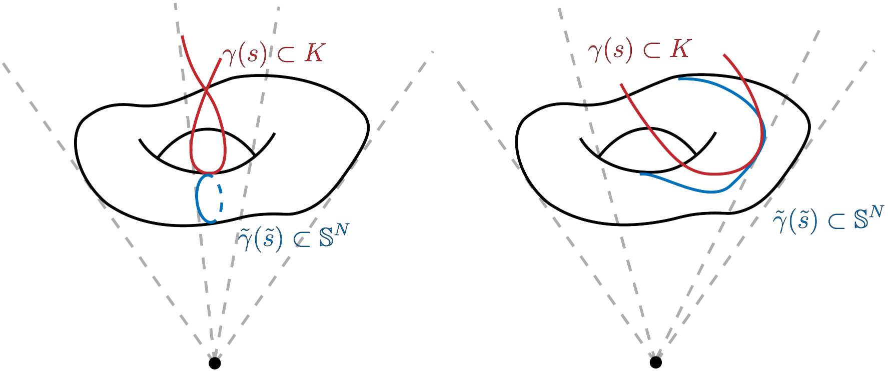
is a geodesic on $K$, with $s$ as its arc-length parameter, $\|\gamma'(s)\| = 1$, touching $\mathbb{S}^N$ at $\tilde{\gamma}(0) = \gamma(0)$.

3) The following limits exist:

$$\lim_{s\to+\infty} \gamma'(s) = \lim_{s\to+\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = \tilde{\gamma}\left(\frac{\pi}{2}\right) \in K,$$

$$\lim_{s\to-\infty} \gamma'(s) = -\lim_{s\to-\infty} \frac{\gamma(s)}{\|\gamma(s)\|} = -\tilde{\gamma}\left(-\frac{\pi}{2}\right) \in K.$$

# Remark


$\gamma(s) \subset K$
$\tilde{\gamma}(\tilde{s}) \subset \mathbb{S}^N$
$\gamma(s) \subset K$
$\tilde{\gamma}(\tilde{s}) \subset \mathbb{S}^N$

## Superintegrability

**Theorem 2**

There exist continuous first integrals $I^1, \ldots, I^{2N+2} : T^1K \to \mathbb{R}$ of the geodesic flow on $T^1K$, which are $C^1$-smooth on $T^1K^+ \cup T^1K^-$, such that the image of the map

$$\mathcal{I} = (I^1, \ldots, I^{2N+2}) : T^1K \to \mathbb{R}^{2N+2}$$

has the property that each point in the image, except the origin in $\mathbb{R}^{2N+2}$, determines a unique geodesic on $K$, while the origin corresponds to all radial geodesics on $K$.

## Proof of Thoerem 2: Construction of First Integrals

- Let us define two subsets of $T^1K$:

$$\begin{aligned}\mathcal{T} &= \left\{(x,v) \in T^1K \mid x \neq kv \text{ for any } k \in \mathbb{R}\right\},\\ \mathcal{S} &= \left\{(x,v) \in T^1K \mid x = kv \text{ for some } k \in \mathbb{R}\right\}.\end{aligned}$$

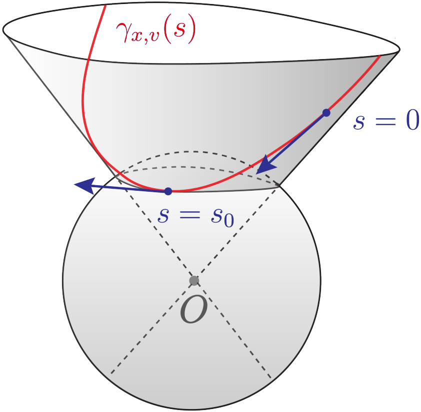
- For $(x,v) \in \mathcal{T}$, let $\gamma_{x,v}(s)$ denote the geodesic with

$$\gamma_{x,v}(0) = x, \quad \gamma'_{x,v}(0) = v.$$

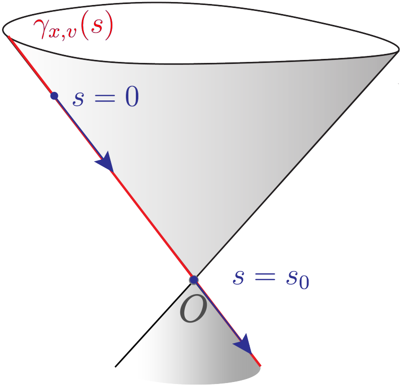
- Any non-radial geodesic $\gamma_{x,v}(s)$ on $K$ touches the sphere

$$\mathbb{S}^N(\sqrt{I}) = \{x \in \mathbb{R}^{N+1} : \|x\|^2 = I\}$$

at a unique point $\gamma_{x,v}(s_0)$, where $s_0 = s_0(x,v)$ is a function depends on $(x,v)$.



$(x, v) \in \mathcal{T}$



$(x, v) \in \mathcal{S}$

- Let us define functions on $\mathcal{T} \subset T^1K$:

$$J_j(x,v) := \gamma^j_{x,v}(s_0(x,v)), \qquad j = 1, \dots, N+1,$$

$$J_{N+1+j}(x,v) := \gamma'^j_{x,v}(s_0(x,v)), \qquad j = 1, \dots, N+1.$$

  Here the superscript $j$ denotes the $j$-th Euclidean coordinate of a vector in $\mathbb{R}^{N+1}$.

- By construction, the functions $J^1, \dots, J^{2N+2}$ are constant along each geodesic trajectory in $\mathcal{T}$ and therefore form first integrals of the system.

## Smooth Integrals on $\mathcal{T}$

### Lemma 3

The functions $J_1, \dots, J_{2N+2}$ are $C^1$-smooth first integrals of the geodesic flow restricted to $\mathcal{T}$. Moreover, the image point

$$\mathcal{J} = (J_1, \dots, J_{2N+2}) \in \mathbb{R}^{2N+2}$$

uniquely determines a non-radial geodesic.

**Remark.** The first $N+1$ integrals $(J_1, \dots, J_{N+1}) = \gamma_{x,v}(s_0)$ can be extended continuously to $\mathcal{S}$, while $(J_{N+2}, \dots, J_{2N+2}) = \gamma'_{x,v}(s_0)$ cannot be extended continuously to $\mathcal{S}$.

## Smooth Integrals on $\mathcal{T} \cup \mathcal{S}$

Let us define

$$\mathcal{I}\colon T^1K \to \mathbb{R}^{2N+2}$$

by

$$\mathcal{I}(x,v) = \begin{cases} e^{-\frac{1}{I^2(x,v)}}\big(\gamma_{x,v}(s_0(x,v)), \gamma'_{x,v}(s_0(x,v))\big), & \text{for } (x,v) \in \mathcal{T}, \\ 0, & \text{for } (x,v) \in \mathcal{S}. \end{cases}$$

Let $I_k$ denote the $j$-th Euclidean coordinate of $\mathcal{I}(x,v)$.

**Lemma 4**

The functions $I_1, \dots, I_{2N+2}$ are first integrals of the geodesic flow on $T^1K$ with the following properties:

1) They are continuous on the entire unit tangent bundle $T^1K$;
2) They are $C^1$-smooth on the smooth part $T^1K^+ \cup T^1K^-$ of $T^1K$;
3) The map $\mathcal{I} = (I_1, \dots, I_{2N+2})$ uniquely determines any non-radial geodesic (where $\mathcal{I} \neq 0$), while mapping all radial geodesics to the origin $0 \in \mathbb{R}^{2N+2}$.

## Complete Integrability

**Theorem 3**

The geodesic flow restricted to the open dense subset $\mathcal{M} \subset T^*K^+$ is completely integrable; that is, there exist $n+1$ smooth first integrals $\mathcal{F}_0 = H, \mathcal{F}_1, \ldots, \mathcal{F}_n$ on $\mathcal{M}$ that are pairwise in involution, i.e., $\{\mathcal{F}_i, \mathcal{F}_j\} = 0$, and functionally independent almost everywhere.

## Complete Integrability

- Let $(t, z)$ be local coordinates on $K^+$, where $t$ is the radial coordinate and $z$ represents coordinates on $\Sigma$. Let $(p_0, p)$ be the corresponding dual coordinates on cotangent space.
- The cone $K^+$ carries the metric

$$g = dt^2 + t^2 \sum_{i,j} g_{ij} dz^i dz^j .$$

- The geodesic flow is generated by the Hamiltonian

$$H(t, z; p_0, p) = \frac{1}{2} p_0^2 + \frac{1}{2t^2} \sum_{i,j} g^{ij}(z) p_i p_j$$

  obtained via the *Legendre transform*.
- We restrict to the open dense subset

$$\mathcal{M} = \{(t, z; p_0, p) \in T^* K^+ \mid p \neq 0\},$$

  which corresponds to non-radial geodesics.

## Construction of Commuting First Integrals

- Extend the integrals $J_k$, $k = 1, \ldots, N+1$, from unit-speed geodesics to all nonzero velocities by requiring zero-degree homogeneity in $v$:

$$\tilde{J}_k(x, v) := J_k\left(x, \frac{v}{\|v\|}\right), \qquad v \neq 0.$$

- Modify the first integrals $\tilde{J}_k(x, v)$ by $\sqrt{I}$,

$$\tilde{F}_k = \frac{\tilde{J}_k}{\sqrt{I}}, \qquad k = 1, \ldots, N+1.$$

- Via the Legendre transform, these functions define smooth first integrals

$$F_k \colon \mathcal{M} \subset T^*K^+ \to \mathbb{R}.$$

- The integrals $F_k$ are in involution and satisfy

$$\operatorname{rank}(F_1, \ldots, F_{N+1}) = \dim \Sigma = n.$$

Thak you very much!