Connection problem for Painlevé II tau function

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Painlevé equations [P. Painlevé & B. Gambier, 1900–1910]:

$$\mathrm{PVI}: \quad q'' = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)(q')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)q' + \\ + \frac{2q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{q^2} + \frac{\gamma(t-1)}{(q-1)^2} + \frac{\delta t(t-1)}{(q-t)^2}\right)$$

$$\mathrm{PV}: \quad q'' = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(q')^2 - \frac{q'}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \\ + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}$$

$$\mathrm{PIV}: \quad q'' = \frac{(q')^2}{2q} + \frac{3}{2}q^3 + 4tq^2 + 2\left(t^2 - \alpha\right)q + \frac{\beta}{q},$$

$$\mathrm{PIII}: \quad q'' = \frac{(q')^2}{q} - \frac{q'}{t} + \frac{\alpha q^2 + \beta}{t} + \gamma q^3 + \frac{\delta}{q},$$

$$\mathrm{PII}: \quad \boxed{q'' = 2q^3 + tq - \hat{\theta}}$$

$\longleftarrow$ equation we will be interested in

$$\mathrm{PI}: \quad q'' = 6q^2 + t.$$

	PVI	PV	PIII_1	PIII_2	PIII_3	PIV	PII	PI
$\sharp$(parameters)	4	3	2	1	0	2	1	0

Classification of nonlinear ODEs of the form

$$q'' = R\left(q', q, t\right),$$

with R rational in q', q, t, such that the general solution $q\left(t; C_1, C_2\right)$ does not have movable singularities other than poles (Painlevé property)

PAUL PAINLEVÉ

Example 1 (logarithmic branch point)

$$q'' = q'^2 \qquad \Longrightarrow \qquad q\left(t\right) = -\ln\left(t - C_1\right) + C_2$$

Example 2 (Weierstrass $\wp$, double pole)

$$q'' = 6q^2 - \frac{g_2}{2} \qquad \Longrightarrow \qquad q\left(t\right) = \wp\left(t - C_1 \middle| g_2, C_2\right)$$

Example 3 (Painlevé II)

$$q'' = 2q^3 + tq - \hat{\theta} \qquad \Longrightarrow$$

$$q\left(t\right) = \begin{cases} C_1 + C_2\left(t - C_1\right) + \ldots, & \text{or} \\ \dfrac{1}{t - C_1} - \dfrac{C_1}{6}\left(t - C_1\right) + \dfrac{\hat{\theta}}{4}\left(t - C_1\right)^2 + C_2\left(t - C_1\right)^3 + \ldots \end{cases}$$

Painlevé property means that PII transcendent $q\left(t\right)$ is meromorphic in the entire complex plane with only simple poles.

> Certainly, an equation is "more integrated" when we know that its solutions are entire or meromorphic functions as compared to when we can only apply the Cauchy method or Picard successive approximations... But this should not make us forget the difference between the existence of an expression and effectively knowing what it is. When Jacobi has expressed in terms of theta functions the inverse function of the elliptic integral of the first kind, that is the solution of a first order differential equation, he has not contented himself with showing that the introduced functions were entire: he has given a precise expression for them via series *with explicitly known successive terms*... We must be able to do the same with Painlevé transcendents, and we should have at our disposal all ingredients necessary to conveniently describe their properties.
>
> Jacques Hadamard, *L'oeuvre scientifique de Paul Painlevé*, (1934)

By now, this description is complete for Painlevé VI and partially accomplished for Painlevé V and III (explicit convergent series representations as $t \to 0$ available, asymptotic series as $t \to \infty$ missing).

Tau function:

Let us define the PII tau function by

$$\frac{d\ln\tau}{dt} = \sigma = \frac{q'^2 - q^4 - tq^2}{2} + \theta q - \frac{t^2}{8}, \qquad \theta = \hat{\theta} + \frac{1}{2}$$

- σ-form of Painlevé II:

$$\left(\sigma''\right)^2 = 2\sigma'\left(\sigma - t\sigma'\right) - 4\left(\sigma'\right)^3 + \frac{\theta^2}{4}$$

- q can be reconstructed from σ using $q = -\dfrac{2\sigma'' + \theta}{4\sigma'}$
- the symmetry $\theta \mapsto -\theta$ of σ-equation is not a symmetry of the q-equation $q'' = 2q^3 + tq - \theta + \frac{1}{2} \implies$ Bäcklund transformations $\hat{\theta} \mapsto \hat{\theta} + 1$
- the tau function is holomorphic in the entire complex t-plane, with simple zeros at poles of q:

$$\tau(t) = \mathcal{N} \cdot (t - C_1)\left[1 - \left(5C_2 + \frac{C_1{}^2}{36}\right)(t - C_1) + \ldots\right]$$

- there is a $\mathbb{Z}_3$-symmetry $t \mapsto \xi t$, $q \mapsto \xi^2 q$ with $\xi^3 = 1$, meaning that if $q(t)$ solves Painlevé II, then $e^{\pm\frac{2\pi i}{3}} q\left(e^{\pm\frac{2\pi i}{3}} t\right)$ are also solutions; likewise, if $\tau(t)$ is a PII tau function, then so are $\tau\left(e^{\pm\frac{2\pi i}{3}} t\right)$

Why tau functions?

- In applications, the quantities of interest are usually tau functions. E.g.

$$\text{Tracy-Widom probability distribution} = \tau_{\mathrm{PII}}(t)$$

  for $\hat{\theta} = -\frac{1}{2}$ and special initial conditions.

- Any PII tau function gives a KdV tau function by

$$\tau_{\mathrm{KdV}}(x,t) = e^{\frac{x^3}{72t}} \tau_{\mathrm{PII}}\left(\frac{x}{\sqrt[3]{3t}}\right)$$

  That is, $u(x,t) = -2(\ln \tau_{\mathrm{KdV}})_{xx}$ solves $u_t - 6uu_x + u_{xxx} = 0$. Solutions not fast decaying as $x \to \pm\infty$!

- (Generic) Painlevé tau functions are also directly related to
  - Virasoro conformal blocks
  - partition functions of supersymmetric gauge theories
  - topological recursion partition functions
  - ...
    [Gamayun, Iorgov, OL, Bershtein, Schechkin, Nagoya, Gavrylenko, Iwaki, Nekrasov, Bonelli, Tanzini, Maruyoshi, Sciarappa,...]

Intermediate summary:

- Painlevé II equation $q'' = 2q^3 + tq - \hat{\theta}$
- $\mathbb{Z}_3$-symmetry, Bäcklund transformations
- preferred quantity is the tau function defined by $\frac{d}{dt} \ln \tau = F(q, q', t)$ up to a multiplicative constant
- $q(t)$ meromorphic with simple poles, $\tau(t)$ holomorphic with simple zeros

Asymptotics

The rays $\arg t = 0, \pm\frac{2\pi}{3}$

To describe the asymptotics as $t \to +\infty$, consider the ansatz

$$q(t) \simeq i\sqrt{2t} \sum_{k\in\mathbb{Z}} \sum_{l=0}^{\infty} q_{k,l} r^{-\frac{|k|}{2}-l-k\gamma} e^{\frac{ikr}{6}}, \qquad r = 4\sqrt{2}|t|^{\frac{3}{2}}.$$

Plugging it into PII (and assuming that $|\Re\gamma| < \frac{1}{2}$) yields an overdetermined system of recurrence equations for the coefficients $\{q_{k,l}\}$.

- The leading asymptotic term corresponding to $(k,l) = (0,0)$ satisfies $q_{0,0}^2 = \frac{1}{4}$. We accordingly set

$$q_{0,0} = \frac{\epsilon}{2}, \qquad \epsilon = \pm 1.$$

  Different values of ϵ lead to different asymptotic behaviors of $q(t)$.

- All other coefficients $q_{k,l}$, as well as parameter γ, can be expressed recursively in terms of $q_{\pm1,0}$. In particular,

$$\begin{aligned}
\gamma &= 2iq_{1,0}q_{-1,0} + \epsilon\hat{\theta},\\
\epsilon q_{0,1} &= i\left(3\gamma - \epsilon\hat{\theta}\right), \qquad q_{\pm k,0} = \epsilon^{k-1} q_{\pm1,0}^{k}, \quad k > 0,\\
q_{\pm1,1} &= \mp\frac{i}{12}\left(102\gamma^2 \pm 24\gamma + 17 - 18\hat{\theta}^2 \pm 12\epsilon\hat{\theta}\right) q_{\pm1,0},\\
\epsilon q_{0,2} &= \frac{51\gamma^2}{2} - 9\epsilon\hat{\theta}\gamma - \frac{9\hat{\theta}^2}{2} + 2,\\
&\ldots \qquad \ldots \qquad \ldots
\end{aligned}$$

- The coefficients $q_{\pm1,0}$ therefore parameterize initial conditions for generic PII transcendent. Each of three rays is described by its own pair $(q_{1,0}, q_{-1,0})$.

The rays $\arg t = \pi, \pm\frac{\pi}{3}$

As $t \to -\infty$, $q(t)$ admits an expansion of the form

$$q(t) \simeq 2i\sqrt{|t|}\sum_{k\in\mathbb{Z}}\sum_{l=0}^{\infty}\tilde{q}_{k,l}\, r^{-\frac{|k|}{2}-l-\frac{k\tilde{\gamma}}{2}}\, e^{\frac{ikr}{12}}, \qquad r = 8\,|t|^{\frac{3}{2}},$$

with $q_{k,0} = 0$ for $k \in 2\mathbb{Z}$. This constraint implies in particular that the leading asymptotics of $q(t)$ is determined by the terms with $(k,l) = (\pm1, 0)$. All other coefficients can be computed recursively in terms of $\tilde{q}_{\pm1,0}$. One has, e.g.

$$\tilde{\gamma} = -2i\tilde{q}_{1,0}\tilde{q}_{-1,0},$$

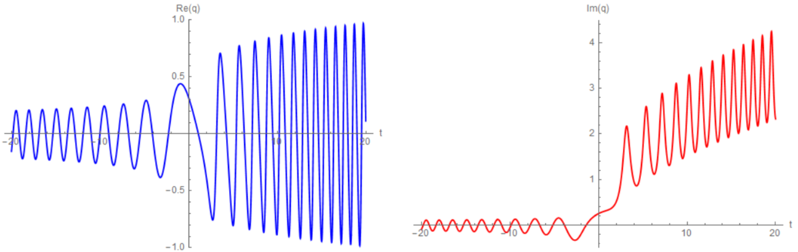
$$\tilde{q}_{0,1} = 4i\hat{\theta}, \qquad \tilde{q}_{\pm(2k+1),0} = \tilde{q}_{\pm1,0}^{2k+1}, \qquad k \geq 0,$$

$$\tilde{q}_{\pm1,1} = \mp\frac{i}{12}\left(51\tilde{\gamma}^2 \pm 36\tilde{\gamma} + 10 - 192\,\hat{\theta}^2\right),$$

$$\tilde{q}_{0,2} = 96\,\hat{\theta}\tilde{\gamma}, \qquad \tilde{q}_{\pm2,1} = 32i\,\hat{\theta}\,\tilde{q}_{\pm1,0}^2,$$

$$\ldots \qquad \ldots \qquad \ldots$$

- Again, each of the rays is characterized by its own pair $(\tilde{q}_{1,0}, \tilde{q}_{-1,0})$ which can be seen as defining the initial conditions for Painlevé II



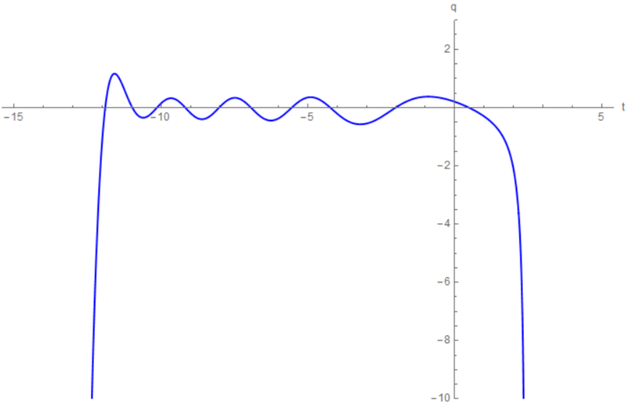
Numerical solution of PII with $\hat{\theta} = 0.1 + 0.2i$, $q(0) = 0.32 + 0.25i$, $q'(0) = -0.23 + 0.1i$

$$q(t \to +\infty) \simeq i\sqrt{2t}\left[\frac{\epsilon}{2} + q_{1,0}\left(4\sqrt{2}t^{\frac{3}{2}}\right)^{-\frac{1}{2}-\gamma} e^{\frac{2i\sqrt{2}}{3}t^{\frac{3}{2}}} + q_{-1,0}\left(4\sqrt{2}t^{\frac{3}{2}}\right)^{-\frac{1}{2}+\gamma} e^{-\frac{2i\sqrt{2}}{3}t^{\frac{3}{2}}}\right],$$

$$q(t \to -\infty) \simeq 2i\sqrt{|t|}\left[\tilde{q}_{1,0}\left(8|t|^{\frac{3}{2}}\right)^{-\frac{1}{2}-\frac{\tilde{\gamma}}{2}} e^{\frac{2i}{3}|t|^{\frac{3}{2}}} + \tilde{q}_{-1,0}\left(8|t|^{\frac{3}{2}}\right)^{-\frac{1}{2}+\frac{\tilde{\gamma}}{2}} e^{-\frac{2i}{3}|t|^{\frac{3}{2}}}\right].$$

Recall that $\epsilon = \pm 1$, $\gamma = 2iq_{1,0}q_{-1,0} + \epsilon\hat{\theta}$ and $\tilde{\gamma} = -2i\tilde{q}_{1,0}\tilde{q}_{-1,0}$.

Connection problem: Express the pair of asymptotic parameters $(\tilde{q}_{1,0}, \tilde{q}_{-1,0})$ at $-\infty$ in terms of the corresponding pair $(q_{1,0}, q_{-1,0})$ at $+\infty$. Solved by [Kapaev,92] !



Numerical solution of PII with $\hat{\theta} = 0.3,\ q(0) = 0.2, q'(0) = -0.35$

NB: Real solutions in general have poles on the real axis and correspond to non-generic initial conditions

What about the tau function ?

Expansions on the rays $\arg t = \frac{2\pi p}{3}$ $(p \in \mathbb{Z}/3\mathbb{Z})$ have the form of Zak transform:

$$\tau(t) = \mathcal{N}_{+,p} \sum_{n\in\mathbb{Z}} \mathcal{D}\left(4\sqrt{2}\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}} \,\middle|\, \nu_p + n, \theta\right) e^{2\pi i n \rho_p},$$

with

$$\mathcal{D}(s\,|\,\nu,\theta) = \frac{G\left(1+\nu+\frac{\theta}{2}\right) G\left(1+\nu-\frac{\theta}{2}\right)}{(2\pi)^{\nu}} s^{\frac{\theta^2}{12}-\nu^2} \exp\left\{\frac{\nu s}{6} + \sum_{k=1}^{\infty} \mathcal{D}_k(\nu,\theta)\, s^{-k}\right\},$$

where

$$\mathcal{D}_1(\nu,\theta) = \frac{\nu\left(68\nu^2 - 9\theta^2 + 2\right)}{6},$$

$$\mathcal{D}_2(\nu,\theta) = 125\nu^4 - \frac{43}{2}\theta^2\nu^2 + \frac{11}{48}\theta^4 + 15\nu^2 - \frac{17}{12}\theta^2,$$

$$\ldots \quad \ldots \quad \ldots$$

- Each of three rays is characterized by a pair of asymptotic parameters (ν_p, ρ_p)
- Asymptotic expansions are periodic with respect to ρ_p and quasiperiodic with respect to ν_p; we can in particular choose $-\frac{1}{2} < |\Re\nu| \leq \frac{1}{2}$
- The leading behavior of $\tau(t)$ is then

$$\tau\left(t \to e^{2\pi i p/3}\infty\right) \simeq \text{const} \cdot s^{\frac{\theta^2}{12}-\nu_p^2} e^{\nu_p s/6}, \qquad s = 4\sqrt{2}\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}}$$

Likewise, expansions on the rays $\arg t = \pi + \frac{2\pi p}{3}$ $(p \in \mathbb{Z}/3\mathbb{Z})$:

$$\tau(t) = \mathcal{N}_{-,p} \sum_{n \in \mathbb{Z}} \mathcal{G}\left(8\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}} \,\middle|\, \omega_p + n, \theta\right) e^{2\pi i n \xi_p},$$

with

$$\mathcal{G}(s\,|\,\omega,\theta) = \frac{G(1+\omega)}{(2\pi)^{\frac{\omega}{2}}} s^{\frac{\theta^2}{3} - \frac{1}{12} - \frac{\omega^2}{2}} \exp\left\{-\frac{s^2}{1536} + \frac{\omega s}{12} + \sum_{k=1}^{\infty} \mathcal{G}_k(\omega,\theta)\, s^{-k}\right\},$$

where the first coefficients are given by

$$\mathcal{G}_1(\omega,\theta) = \frac{\omega\left(34\,\omega^2 - 96\,\theta^2 + 31\right)}{6},$$

$$\mathcal{G}_2(\omega,\theta) = \frac{375\,\omega^4 - 3\left(832\,\theta^2 - 293\right)\omega^2 + 4\left(4\theta^2 - 1\right)\left(4\theta^2 - 9\right)}{6},$$

$$\ldots \quad \ldots \quad \ldots$$

- Each of three rays is characterized by a pair of asymptotic parameters (ω_p, ξ_p)
- Asymptotic expansions are periodic with respect to ξ_p and quasiperiodic with respect to ω_p; we choose $-\frac{1}{2} < |\Re\xi| \leq \frac{1}{2}$
- The leading behavior of $\tau(t)$ is then given by

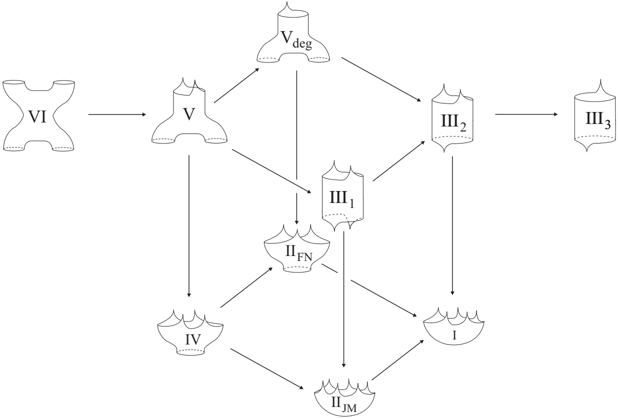
$$\tau\left(t \to -e^{2\pi i p/3}\infty\right) \simeq \text{const} \cdot s^{\frac{\theta^2}{3} - \frac{1}{12} - \frac{\omega_p^2}{2}} e^{-\frac{s^2}{1536} + \frac{\omega_p s}{12}}, \qquad s = 8\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}}$$

Intermediate summary:

- To solve the connection problem for $q(t)$, it suffices to relate pairs of parameters (ν_p, ρ_p), (ω_p, ξ_p), $p \in \mathbb{Z}/3\mathbb{Z}$ characterizing the asymptotics of Painlevé II tau function along six special rays as $t \to \infty$
- The answer can in principle be extracted from [Kapaev,92]
- For now, we did not discuss constant prefactors in the asymptotics of tau functions; however, this is actually the main question we are interested in!

Monodromy

Geometric Painlevé confluence diagram [Chekhov, Mazzocco, Rubtsov, '15]:



- each of Painlevé equations can be seen as equation of monodromy preserving deformations of a class of rank 2 linear systems on the Riemann sphere
- holes correspond to singularities, the number of cusps = 2×Poincaré rank
- E.g. Painlevé VI $\Longrightarrow$ Fuchsian systems with 4 poles

For Painlevé II, we have (at least) two options:

- A single irregular singular point on $\mathbb{CP}^1$ of Poincaré rank 3 (Jimbo-Miwa Lax pair)
- One Fuchsian singular point (simple pole) and one irregular singularity of Poincaré rank $\frac{3}{2}$ (Flaschka-Newell Lax pair)

Let us stick to the 2nd choice, and place the singular points respectively at 0 and ∞

$$\partial_z \Psi = A(z)\,\Psi, \qquad A(z) = A_1 z + A_0 + A_{-1} z^{-1},$$

- 2×2 matrices $A_{\pm 1,0}$ can be assumed traceless
- The rank $\frac{3}{2}$ singularity at ∞ means that A_1 is non-diagonalizable. Using constant gauge transformations, bring it to the form $A_1 = \begin{pmatrix} 0 & -2i \\ 0 & 0 \end{pmatrix}$
- Using upper triangular constant gauge transformations and rescaling ξ, one can further set
$$A_0 = \begin{pmatrix} 0 & \alpha \\ -2i & 0 \end{pmatrix}, \qquad A_{-1} = \begin{pmatrix} \beta & \gamma \\ \delta & -\beta \end{pmatrix}.$$
- Parameterize the remaining four quantities $\alpha, \beta, \gamma, \delta$ it terms of $p, q, t, \hat{\theta}$ as
$$\alpha = i\left(p + q^2 - \frac{t}{2}\right), \qquad \beta = pq + q^3 + \frac{qt}{2} + \frac{1}{4} - \frac{\hat{\theta}}{2},$$
$$\gamma = -iq\left(pq + q^3 + \frac{qt}{2} + \frac{1}{2} - \hat{\theta}\right), \qquad \delta = -i\left(p + q^2 + \frac{t}{2}\right).$$
- The eigenvalues of A_{-1} are given by $\left\{\frac{2\hat{\theta}-1}{4}, -\frac{2\hat{\theta}-1}{4}\right\}$

- Monodromy preserving deformation of $\partial_z \Phi = A(z)\,\Phi$ yields Painlevé II
- Monodromy data consist of three Stokes multipliers $s_1, s_2, s_3 \in \mathbb{C}$ associated to irregular singularity at $z = \infty$ and local monodromy exponent $\hat{\theta} \in \mathbb{C}$ of the Fuchsian singular point at $z = 0$. They satisfy one constraint

$$W = 0, \qquad W = s_1 s_2 s_3 - s_1 - s_2 - s_3 - 2\sin\pi\hat{\theta}.$$

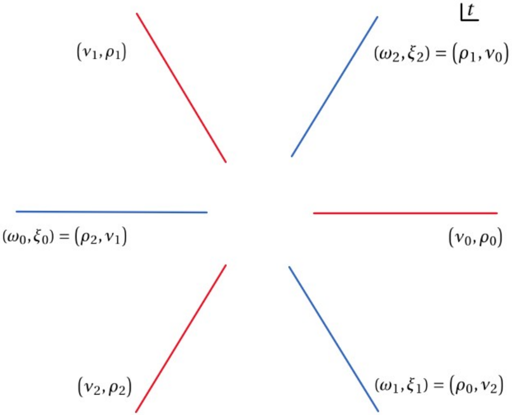
  The Painlevé II space of monodromy data is thus an affine cubic surface $\mathcal{M}$ defined by zero locus of W.
- To any solution of PII thus corresponds a unique triple $(s_1, s_2, s_3) \in \mathcal{M}$. Conversely, to any generic point in $\mathcal{M}$ corresponds a unique solution of PII. The non-generic points of $\mathcal{M}$ are given by

$$s_1 = s_2 = s_3 = (-1)^n, \qquad \hat{\theta} = n - \tfrac{1}{2}, \qquad n \in \mathbb{Z}.$$

- Any triple $(s_1, s_2, s_3) \in \mathcal{M}$ can be seen as a pair of Painlevé II integrals of motion
- The key idea of solution of the connection problem is to relate different pairs of asymptotic parameters through an intermediate object — monodromy

Solution (essentially a rewrite of [Kapaev,92]):

$$e^{2\pi i\nu_2} = e^{2\pi i\xi_1} = s_1\,, \qquad e^{2\pi i\rho_1} = e^{2\pi i\omega_2} = 1 - s_2 s_3\,,$$
$$e^{2\pi i\nu_0} = e^{2\pi i\xi_2} = s_2\,, \qquad e^{2\pi i\rho_2} = e^{2\pi i\omega_0} = 1 - s_1 s_3\,,$$
$$e^{2\pi i\nu_1} = e^{2\pi i\xi_0} = s_3\,, \qquad e^{2\pi i\rho_0} = e^{2\pi i\omega_1} = 1 - s_1 s_2\,.$$



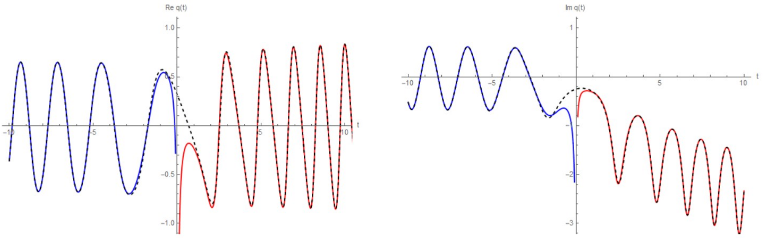
Numerical check

- We found it most convenient to identify the two monodromy parameters from the leading asymptotic behavior of the Painlevé II σ-function on a pair of rays, e.g., as $t \to \pm\infty$. In this case, whenever $\Re\nu_0, \Re\omega_0 \neq \frac{1}{2} \mod \mathbb{Z}$, we have

$$\nu_0 = \lim_{t\to+\infty} \frac{\sigma(t)}{i\sqrt{2t}}, \qquad \omega_0 = -\lim_{t\to-\infty} \frac{\sigma(t) + \frac{t^2}{8}}{i\sqrt{-t}}.$$

- These limiting values thus fix s_2 and the product $s_1 s_3$. One can then compute s_1 and s_3 using the monodromy condition $s_1 s_2 s_3 - s_1 - s_2 - s_3 + 2\cos\pi\theta = 0$. In general, it yields two pairs (s_1, s_3) related by the exchange $s_1 \leftrightarrow s_3$. The correct choice can be easily identified numerically.

Example:



$$\hat{\theta} = 0.1 + 0.2i, \qquad q(0) = 0.32 - 0.25i, \qquad q'(0) = -0.43 + 0.1i,$$

$$\nu_0 \approx -0.369 - 0.0792i, \qquad \omega_0 \approx 0.166 + 0.0613i,$$

$$s_1 \approx -0.994 - 0.870i, \qquad s_2 \approx -1.115 - 1.209i, \qquad s_3 \approx -0.0816 + 0.664i,$$

$$\rho_0 \approx -0.185 - 0.137i, \qquad \xi_0 \approx 0.269 + 0.0640i.$$

Connection coefficients for tau function

Recall the asymptotic expansions of $\tau(t)$.

- On the rays $\arg t = \frac{2\pi p}{3}$ $(p \in \mathbb{Z}/3\mathbb{Z})$:

$$\tau(t) = \mathcal{N}_{+,p} \sum_{n\in\mathbb{Z}} \mathcal{D}\left(4\sqrt{2}\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}} \,\middle|\, \nu_p + n, \theta\right) e^{2\pi i n \rho_p}$$

- On the rays $\arg t = \pi + \frac{2\pi p}{3}$ $(p \in \mathbb{Z}/3\mathbb{Z})$:

$$\tau(t) = \mathcal{N}_{-,p} \sum_{n\in\mathbb{Z}} \mathcal{G}\left(8\, e^{\frac{i\pi}{2}} |t|^{\frac{3}{2}} \,\middle|\, \omega_p + n, \theta\right) e^{2\pi i n \xi_p}.$$

The tau function is defined by $\frac{d \ln \tau}{dt} = \frac{q'^2 - q^4 - tq^2}{2} + \theta q - \frac{t^2}{8}$ only up to a constant normalization factor. The individual prefactors $\mathcal{N}_{\pm,p}$ depend on this normalization. However, their ratios

$$\Upsilon_{\epsilon\epsilon',pp'} = \frac{\mathcal{N}_{\epsilon,p}}{\mathcal{N}_{\epsilon',p'}}, \qquad \epsilon, \epsilon' = \pm, \quad p, p' \in \mathbb{Z}/3\mathbb{Z}.$$

are uniquely determined by the initial conditions/monodromy.

Goal: To find these connection coefficients explicitly.

- Thanks to $\mathbb{Z}_3$-symmetry, it suffices to compute two of them:

$$\Upsilon_{\Ldsh}\left(\nu_0, \omega_1 \,|\, \theta\right) = \Upsilon_{+-,01}, \qquad \Upsilon_{\Lsh}\left(\nu_0, \omega_2 \,|\, \theta\right) = \Upsilon_{+-,02}.$$

Conjecture: We have

$$\boxed{\begin{aligned}
\Upsilon_{\Ldsh}\left(\nu_0, \omega_1 \,|\, \theta\right) &= (2\pi)^{\omega_1}\, \hat{G}\left(-\omega_1\right) e^{\frac{\pi i \omega_1^2}{2} - 2\pi i \nu_0 \omega_1} 2^{-\frac{5\theta^2}{3}} e^{-\frac{i\pi\theta^2}{6}} \chi, \\
\Upsilon_{\Lsh}\left(\nu_0, \omega_2 \,|\, \theta\right) &= \frac{\hat{G}\left(\nu_0 + \frac{\theta}{2}\right) \hat{G}\left(\nu_0 - \frac{\theta}{2}\right)}{(2\pi)^{2\nu_0}} e^{2\pi i \nu_0 \omega_2 - \pi i \nu_0^2} 2^{-\frac{5\theta^2}{3}} e^{-\frac{i\pi\left(\theta^2 - 1\right)}{12}} \chi,
\end{aligned}}$$

where $\hat{G}(z) = \dfrac{G(1+z)}{G(1-z)}$, $G(z)$ denotes the Barnes G-function, and $\chi = 2^{\frac{1}{3}} e^{\zeta'(-1)}$.

Shortcut "derivation":

- Zak transform structure of tau function expansions implies quasiperiodicity relations

$$\begin{cases} \dfrac{\Upsilon_{\hookleftarrow}(\nu_0+1,\omega_1)}{\Upsilon_{\hookleftarrow}^{\mathcal{T}}(\nu_0,\omega_1)} = e^{-2\pi i\rho_0} = e^{-2\pi i\omega_1}, \\ \dfrac{\Upsilon_{\hookleftarrow}(\nu_0,\omega_1+1)}{\Upsilon_{\hookleftarrow}(\nu_0,\omega_1)} = e^{2\pi i\xi_1} = e^{-2\pi i\nu_0}\left(1-e^{2\pi i\omega_1}\right), \end{cases}$$

$$\begin{cases} \dfrac{\Upsilon_{\looparrowleft}(\nu_0+1,\omega_2)}{\Upsilon_{\looparrowleft}(\nu_0,\omega_2)} = e^{-2\pi i\rho_0} = -\dfrac{e^{2\pi i(\omega_2-\nu_0)}}{4\sin\pi\left(\nu_0+\frac{\theta}{2}\right)\sin\pi\left(\nu_0-\frac{\theta}{2}\right)}, \\ \dfrac{\Upsilon_{\looparrowleft}(\nu_0,\omega_2+1)}{\Upsilon_{\looparrowleft}^{\mathcal{T}}(\nu_0,\omega_2)} = e^{2\pi i\xi_2} = e^{2\pi i\nu_0}. \end{cases}$$

- The general solutions of these equations are given by

$$\Upsilon_{\hookleftarrow}(\nu_0,\omega_1) = (2\pi)^{\omega_1}\,\hat{G}(-\omega_1)\,e^{\frac{\pi i\omega_1^2}{2}-2\pi i\nu_0\omega_1}\chi_{\hookleftarrow}(\nu_0,\omega_1),$$

$$\Upsilon_{\looparrowleft}(\nu_0,\omega_2) = \frac{\hat{G}\left(\nu_0+\frac{\theta}{2}\right)\hat{G}\left(\nu_0-\frac{\theta}{2}\right)}{(2\pi)^{2\nu_0}}e^{2\pi i\nu_0\omega_2-\pi i\nu_0^2}\chi_{\looparrowleft}(\nu_0,\omega_2),$$

where $\chi_{\hookleftarrow}(\nu_0,\omega_1)$, $\chi_{\looparrowleft}(\nu_0,\omega_2)$ are 1-periodic in both of their arguments. Let us be brave and assume that $\chi_{\hookleftarrow}$ and $\chi_{\looparrowleft}$ do not depend on ν_0,ω_1,ω_2 at all. They may however depend on θ.

To fix the θ dependence, we use

- $\mathbb{Z}_3$-symmetry applied to completely symmetric monodromy

$$s_1 = s_2 = s_3 = u = -2\cos\frac{\pi\theta}{3} \quad\Longrightarrow\quad e^{2\pi i\nu_k} = e^{2\pi i\xi_k} = u, \quad e^{2\pi i\omega_k} = e^{2\pi i\rho_k} = 1 - u^2.$$

  In this case, $\tau\left(e^{\frac{2\pi i}{3}}t\right) = \zeta\tau(t)$ with $\zeta^3 = 1$.

- Using Yablonskii-Vorobiev solutions of Painlevé II (essentially, polynomial KdV tau functions) one may check that $\zeta = 1$.
- This yields a relation

$$\frac{\chi_{\leftarrow}(\theta)}{\chi_{\hookleftarrow}(\theta)} = e^{\frac{i\pi\left(\theta^2+1\right)}{12}}.$$

- Bäcklund transformations (Toda lattice equation) leading to

$$\frac{\chi_{\hookleftarrow}(\theta+1)\,\chi_{\hookleftarrow}(\theta-1)}{\chi_{\hookleftarrow}(\theta)^2} = 2^{-\frac{10}{3}}e^{-\frac{i\pi}{3}}, \qquad \frac{\chi_{\leftarrow}(\theta+1)\,\chi_{\leftarrow}(\theta-1)}{\chi_{\leftarrow}(\theta)^2} = 2^{-\frac{10}{3}}e^{-\frac{i\pi}{6}}.$$

This fixes $\chi_{\leftarrow}(\theta)$, $\chi_{\hookleftarrow}(\theta)$ up to a 1-periodic function of θ. We are now going to be even more brave and assume that it is constant. Its value can be derived from the known asymptotics of the Tracy-Widom distribution (Airy kernel determinant).

Concluding remarks:

- Work in progress: rigorous proof using the extension of Jimbo-Miwa-Ueno differential to the space of monodromy data [Its, OL, Prokhorov,'16]
- Other Painlevé equations:
  - the answer is known and proved for PVI, PI and PIII(D_8)
  - conjectural answer is available for PV
  - for PIV, PIII(D_6) and PIII(D_7), the problem is completely open...

THANK YOU!