Conformal blocks and associative algebras in logarithmic conformal field theory

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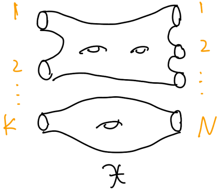
Goal of the talk

- What is conformal field theory (CFT)? What are conformal blocks?
- The factorization property for conformal blocks in the semi-simple setting (i.e. rational CFT).
- Difficulties in the non semi-simple setting (i.e. logarithmic CFT).
- How to overcome the difficulties?

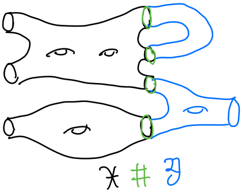
- A $\mathbb{C}$-vector space $\mathbb{A}$ (the space of quantum states with finite energies).
- For each compact Riemann surface $\mathfrak{X}$ with parametrized boundary, divided into two groups (K outgoing and N incoming boundaries)



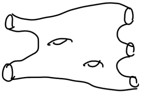
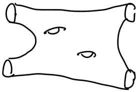
- A linear map (**correlation function**) $\Phi_{\mathfrak{X}} : \mathbb{A}^{\otimes N} \to \overline{\mathbb{A}^{\otimes K}}$ (where $\overline{\mathbb{A}^{\otimes K}}$ is a certain "completion" of $\overline{\mathbb{A}^{\otimes K}}$, e.g. Hilbert space completion in the unitary case).
- $\Phi_{\mathfrak{X}}$ relies real-analytically on $\mathfrak{X}$.

What is Conformal Field Theory (CFT)?

- Composition law: $\Phi_{\mathfrak{X}\#\mathfrak{Y}} = \Phi_{\mathfrak{X}} \circ \Phi_{\mathfrak{Y}}$ (sometimes up to multiplication by $\lambda_{\mathfrak{X},\mathfrak{Y}} \in \mathbb{C}$ depending real-analytically on $\mathfrak{X}, \mathfrak{Y}$).

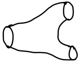


- There is a "vacuum vector" $\mathbf{1}$ such that $\Phi_{\mathfrak{X}}(- \otimes \mathbf{1} \otimes -) = \Phi_{\mathfrak{X}'}$ if

$\mathfrak{X} =$  $\mathfrak{X}' =$ 

What is a Vertex Operator Algebra (VOA)

The rigorous definition of VOA is due to Borcherds and Frenkel-Lepowsky-Meurman in the 1980s. Physically:

- A vertex operator algebra (VOA) is the space $\mathbb{V}$ consisting of all $v \in \mathbb{A}$ such that for each $\mathfrak{X}$, the map $\Phi_{\mathfrak{X}}(- \otimes v \otimes -)$ is holomorphic with respect to the position and the parametrization of the boundary circle for v. (Namely, $\mathbb{V}$ is the space of **chiral fields**.)
- $\mathbb{V}$ form an "algebra" if the "multiplication" is defined by $\Phi_{\Sigma} : \mathbb{V} \otimes \mathbb{V} \to \overline{\mathbb{V}}$ where Σ is a pair of pants 
- The vacuum vector $\mathbf{1}$ can be viewed as the identity.
- $\mathbb{A} \supset \mathbb{V} \otimes \mathbb{V}'$ where $\mathbb{V}'$ is the space of **anti-chiral fields**.
- In this talk we assume $\mathbb{V}' \simeq \mathbb{V}$. Then $\mathbb{A} \in \mathrm{Mod}(\mathbb{V}^{\otimes 2})$.

What is a **rational CFT**?

- $\mathbb{A} = \bigoplus_{i,j} \mathbb{M}_i \otimes_{\mathbb{C}} \mathbb{M}_j$ is a finite direct sum, where $\mathbb{M}_i, \mathbb{M}_j \in \mathrm{Irr}(\mathbb{V})$.
- The restriction of $\Phi_{\mathfrak{X}} : \mathbb{A}^{\otimes N} \to \overline{\mathbb{A}^{\otimes K}}$ to

  $(\mathbb{M}_{i_1} \otimes \mathbb{M}_{j_1}) \otimes \cdots \otimes (\mathbb{M}_{i_N} \otimes \mathbb{M}_{j_N}) \to \overline{(\mathbb{M}_{s_1} \otimes \mathbb{M}_{t_1}) \otimes \cdots \otimes (\mathbb{M}_{s_K} \otimes \mathbb{M}_{t_K})}$

  is a finite sum $\sum_\alpha \varphi_\alpha^\sharp \otimes \psi_\alpha^\sharp$ where

$$\varphi_\alpha^\sharp : \mathbb{M}_{i_1} \otimes \cdots \otimes \mathbb{M}_{i_N} \to \overline{\mathbb{M}_{s_1} \otimes \cdots \otimes \mathbb{M}_{s_K}}$$
$$\psi_\alpha^\sharp : \mathbb{M}_{j_1} \otimes \cdots \otimes \mathbb{M}_{j_N} \to \overline{\mathbb{M}_{t_1} \otimes \cdots \otimes \mathbb{M}_{t_K}}$$

  are also viewed as linear functionals

$$\varphi_\alpha : \mathbb{M}_{i_1} \otimes \cdots \otimes \mathbb{M}_{i_N} \otimes \mathbb{M}_{s_1}^\vee \otimes \cdots \otimes \mathbb{M}_{s_K}^\vee \to \mathbb{C}$$
$$\psi_\alpha : \mathbb{M}_{j_1} \otimes \cdots \otimes \mathbb{M}_{j_N} \otimes \mathbb{M}_{t_1}^\vee \otimes \cdots \otimes \mathbb{M}_{t_K}^\vee \to \mathbb{C}.$$

- The linear functionals φ_α and ψ_α are **conformal blocks** associated to $\mathfrak{X}$ and the conjugate $\mathfrak{X}^\dagger$. They rely holomorphically on $\mathfrak{X}$ and $\mathfrak{X}^\dagger$, respectively.

A slight modification of the geometric setting

- Compact Riemann surfaces with parametrized boundaries $\approx$ Compact Riemann surfaces with marked points and local coordinates.

- RHS: $\mathfrak{X} = (C; x_1, \ldots, x_n; \eta_1, \ldots, \eta_n)$ where $x_1, \ldots, x_n \in C$ are distinct, and η_j is a **local coordinate** at x_j, i.e. a biholomorphism $\eta_j : U_j \xrightarrow{\simeq} \varphi(U_j) \subset \mathbb{C}$ where U_j is a neighborhood of x_j and $\eta_j(x_i) = 0$.
- LHS$\rightsquigarrow$RHS: Attach $\mathbb{D} = \{|z| \leqslant 1\}$ to the holes.
- RHS$\rightsquigarrow$LHS: Remove $B_j = \{x \in C : |\eta_i(x)| \leqslant 1\}$. Then $\eta_j|_{\partial B_j}$ is the parametrization of the j-th boundary ∂B_j.

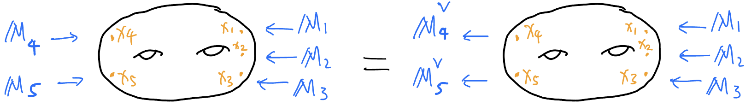
Conformal Blocks (CB)

- Given a VOA $\mathbb{V}$ and $\mathfrak{X} = (C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N)$ and $\mathbb{M}_1, \ldots, \mathbb{M}_N \in \mathrm{Mod}(\mathbb{V})$ associated to $x_1, \ldots, x_N$, a conformal block (CB) is a linear functional

$$\varphi : \mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_N \to \mathbb{C}$$

that is "invariant under the action of $\mathbb{V}$".

- Both figures denote CB of this type:



- Rigorous definition: Tsuchiya-Ueno-Yamada (89), Zhu (94), Frenkel–Ben-Zvi (01).

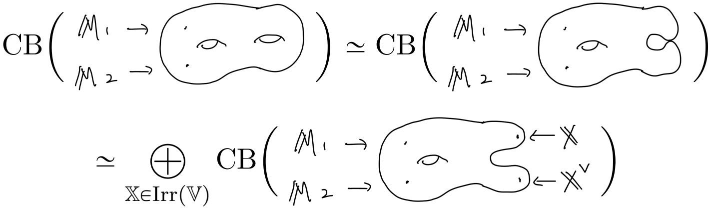
C_2-cofinite Rational VOAs

- Physics: Rational CFT = Math: C_2-cofinite and rational.
- Examples: Affine VOAs of positive integer levels, lattice VOAs, etc. Their tensor products, extensions, relative commutants, fixed point subalgebras under finite (solvable groups), etc.
- C_2-cofinite (Zhu 96): It implies $\dim \mathrm{CB}(\mathfrak{X}, \mathbb{M}_\bullet) < +\infty$ and is a topological invariant of $\mathfrak{X}$, $\mathrm{Mod}(\mathbb{V})$ is a finite abelian category, etc. (In particular, $\#\mathrm{Irr}(\mathbb{V})$ is finite.)
- Rational: $\mathrm{Mod}(\mathbb{V})$ is semi-simple.
- Zhu's algebra $A(\mathbb{V})$ (Zhu 96): It is a unital associative $\mathbb{C}$-algebra.
  - $\mathrm{Irr}(\mathbb{V}) \simeq \mathrm{Irr}(A(\mathbb{V}))$ in general.
  - C_2-cofinite $\Longrightarrow \dim A(\mathbb{V}) < +\infty$.
  - C_2-cofinite and rational $\Longrightarrow \mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Mod}^{\mathrm{L}}(A(\mathbb{V}))$ where $\mathrm{Mod}^{\mathrm{L}}(A(\mathbb{V}))$ is the category of finite-dimensional left $A(\mathbb{V})$-modules.

Conformal blocks for C_2-cofinite rational VOAs

Assume that $\mathbb{V}$ is C_2-cofinite and rational.

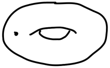
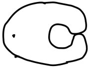
- Corresponding to the composition law $\Phi_{\mathfrak{X}\#\mathfrak{Y}} = \Phi_{\mathfrak{X}} \circ \Phi_{\mathfrak{Y}}$ in the full CFT, we have the factorization property (Tsuchiya-Ueno-Yamada 89, Zhu 96, Huang 05, Nagatomo-Tsuchiya 05, Damiolini-Gibney-Tarasca 19):



Conformal blocks for C_2-cofinite rational VOAs

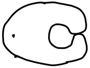
Assume that $\mathbb{V}$ is C_2-cofinite and rational.

- A key special case by Zhu (96) is

$\mathrm{CB}\big(\mathbb{V} \to$  $\big) \simeq \mathrm{CB}\big(\mathbb{V} \to$  $\big)$

$\simeq \mathrm{Span}\{$characters of irreducibles$\}$

- Zhu proved this by proving (rational or C_2-cofinite not needed)

$\mathrm{CB}\big(\mathbb{V} \to$  $\big) \simeq \mathrm{SLF}(A(\mathbb{V}))$

where the space of symmetric linear functionals on a $\mathbb{C}$-algebra A is denoted by

$$\mathrm{SLF}(A) = \{\text{linear } \varphi : A \to \mathbb{C} : \varphi(ab) = \varphi(ba) \text{ if } a, b \in A\}$$

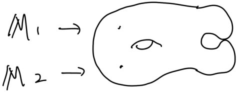
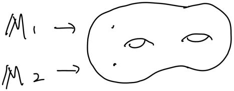
From rational CFT to logarithmic CFT

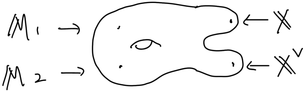
- Factorization is the key property ensuring that conformal blocks satisfy the axioms of a modular functor (Bakalov-Kirillov 01, Andersen-Ueno 07, Damiolini-Woike 25).
- Modular functor + rigidity of $\mathrm{Mod}(\mathbb{V})$ (Huang 08, Etingof-Penneys 24) implies that the spaces of conformal blocks form a 3d topological quantum field theory.
    - Example: If $\mathbb{V}$=affine VOAs of positive integer levels, the 3d TQFT=Chern-Simons TQFT. (Witten 89, Reshetikhin-Turaev 91)
- Resent research has been focusing on non-semisimple 3d TQFT, initiated by Lyubashenko in 1994. They come from **logarithmic CFT** of finite-type, the VOA $\mathbb{V}$ of which is C_2-cofinite but not rational.
    - Examples: Triplet algebras, symplectic fermions, etc.

What fails for C_2-cofinite VOAs that are not rational

From now on, we always assume that $\mathbb{V}$ is C_2-cofinite, but not (necessarily) rational. (Thus $\mathrm{Mod}(\mathbb{V})$ is finite abelian, but not necessarily semi-simple.)

- In general, we only have

$\dim \mathrm{CB}\Big($  $\Big) \geqslant \dim \mathrm{CB}\Big($  $\Big)$

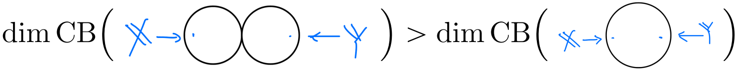
$\geqslant \sum_{\mathbb{X}\in\mathrm{Irr}(\mathbb{V})} \dim \mathrm{CB}\Big($  $\Big)$

  where the two inequalities could be strict.

- The second strict inequality is easy. The first strict inequality is more difficult to obtain, and it implies that the spaces of CB form a vector bundle on $\mathcal{M}_{g,n}$, but not on $\overline{\mathcal{M}}_{g,n}$.

What fails for C_2-cofinite VOAs that are not rational

- In the rational case, the fact that spaces of CB form a vector bundle on $\overline{\mathcal{M}}_{g,n}$ but not just on $\mathcal{M}_{g,n}$ is crucial for proving the factorization. In the logarithmic case, conformal blocks on nodal curves are not as effective as in the rational case.
- The first inequality was proved by Hao Zhang in arXiv: 2509.07720 in genus-0 for some $\mathbb{X}, \mathbb{Y} \in \mathrm{Mod}(\mathbb{V})$:



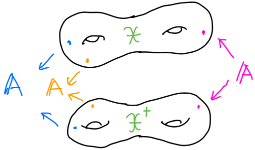
  where $\mathbb{V}$ is a triplet algebra or a symplectic fermion.
    - $\mathbb{X}$ is projective indecomposable. $\mathbb{Y}$ is either projective indecomposible, or irreducible.

What fails for C_2-cofinite VOAs that are not rational

- Due to the close relationship between nodal CB and $A(\mathbb{V})$ (Zhu 96, Heluani-van Ekeren 18, Damolini-Gibney-Krashen 22-24, etc.), $A(\mathbb{V})$ is not enough for the study of CB in logarithmic CFT.

Another difficulty:

- The $\mathbb{V}^{\otimes 2}$-module $\mathbb{A}$ does not decompose as $\bigoplus_{i,j} \mathbb{M}_i \otimes \mathbb{M}_j$. Thus, $\Phi_{\mathfrak{X}}$ does not factor as $\sum_\alpha \varphi_\alpha^\sharp \otimes \psi_\alpha^\sharp$.
- How to overcome: View $\Phi_{\mathfrak{X}}$ directly as a conformal block associated to $\mathfrak{X} \sqcup \mathfrak{X}^\dagger$ and $\mathbb{A}, \ldots, \mathbb{A}$, where $\mathfrak{X}^\dagger$ is the conjugate manifold of $\mathfrak{X}$.

$\Phi_{\mathfrak{X}} \in \mathrm{CB}\Bigg($  $\Bigg)$

Lesson: *Study CB for $\mathbb{V}^{\otimes N}$-modules not necessarily of the form $\mathbb{M}_1 \otimes \cdots \otimes \mathbb{M}_N$ (or a direct sum of those of this form)!*

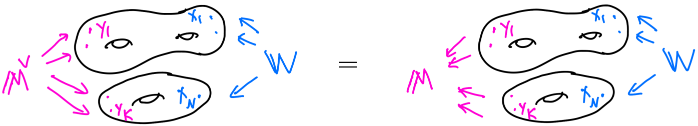
Conformal blocks for C_2-cofinite VOAs

- Given $\mathfrak{X} = (C; x_1, \dots, x_N; \eta_1, \dots, \eta_N)$ and $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ associated in order to $x_1, \dots, x_N$, a conformal block (CB) is a linear functional

$$\varphi : \mathbb{W} \to \mathbb{C}$$

that is "invariant under the action of $\mathbb{V}$".

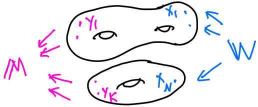
- The contravariant functor $\mathbb{W} \mapsto \mathrm{CB}(\mathfrak{X}, \mathbb{W})$ is left exact.
- Suppose $\mathfrak{X}$ has $N + K$ marked points, divided into N inputs and K outputs. Let $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N}), \mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$, then



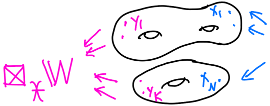
both denote CB $\varphi : \mathbb{M}^\vee \otimes \mathbb{W} \to \mathbb{C}$, equivalently $\varphi^\sharp : \mathbb{W} \to \overline{\mathbb{M}}$.

The fusion product $\boxtimes_{\mathfrak{X}}\mathbb{W}$

- Any left exact functor from a finite $\mathbb{C}$-linear category to $\mathcal{V}ect_{\mathbb{C}}$ is representable (Douglas-SchommerPries-Snyder 19).
- Thus, for each $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$, there is an $\mathbb{M}$-natural isomorphism

$\mathrm{CB}\Bigg($  $\Bigg) \simeq \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \mathbb{M})$

for some $\boxtimes_{\mathfrak{X}}\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes K})$, called the **fusion product** of $\mathbb{W}$ along $\mathfrak{X}$.
- The **canonical conformal block** denotes the element

$\beth_{\mathfrak{X}} \in \mathrm{CB}\Bigg($  $\Bigg)$

corresponding to $\mathrm{id}_{\boxtimes_{\mathfrak{X}}\mathbb{W}} \in \mathrm{Hom}_{\mathbb{V}^{\otimes K}}(\boxtimes_{\mathfrak{X}}\mathbb{W}, \boxtimes_{\mathfrak{X}}\mathbb{W})$ via the above $\simeq$.

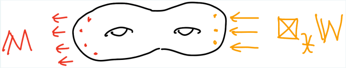
The Sewing-Factorization (SF) theorem

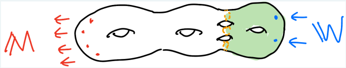
View $\gimel_{\mathfrak{X}} \in \mathrm{CB}\Big($ 

$\Big)$ as a linear map $\gimel^\sharp_{\mathfrak{X}} : \mathbb{W} \to \overline{\boxtimes_{\mathfrak{X}}(\mathbb{W})}$.

Theorem (Sewing-Factorization (SF) theorem G.-Zhang, arXiv:2503.23995)

Assuming $\mathbb{V}$ is C_2-cofinite, we have a natural (in $\mathbb{M}$ and $\mathbb{W}$) linear isomorphism (called the **sewing-factorization (SF) isomorphism***)*

$\mathrm{CB}\Big($ 

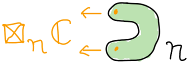
$\Big) \xrightarrow{\simeq} \mathrm{CB}\Big($ 

$\Big)$

$$\varphi^\sharp \mapsto \varphi^\sharp \circ \gimel^\sharp_{\mathfrak{X}}$$

- To show that the above composition is well-defined, we need to check certain convergence property. This is done in our paper arXiv:2411.07707.

The Sewing-Factorization (SF) theorem

View the complex field $\mathbb{C}$ as an object of $\mathrm{Mod}(\mathbb{V}^{\otimes 0})$, and let $\boxtimes_{\mathfrak{N}}\mathbb{C}$ the fusion product of $\mathbb{C}$ along a sphere $\mathfrak{N}$ with 0 inputs and 2 outputs 

Corollary

Assuming $\mathbb{V}$ is C_2-cofinite, we have an $\mathbb{M}$-natural linear isomorphism

$\mathrm{CB}\Big($  $\Big) \simeq \mathrm{CB}\Big($  $\Big)$

defined by the SF isomorphism.

- In fact, $\boxtimes_{\mathfrak{N}}\mathbb{C}$ is the Cardy state space, i.e., the state space for the Cardy-type full CFT. Its canonical conformal block is the Cardy state.

$\boxtimes_{\mathfrak{N}}\mathbb{C} = \int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$

Recall that $\mathbb{V}$ is assumed to be C_2-cofinite.

- By a result of Fuchs-Schaumann-Schweigert 16, we have a canonical isomorphism

$$\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes_{\mathbb{C}}\mathbb{X}^\vee$$

  where the RHS is the end of the bifunctor

$$\mathrm{Mod}(\mathbb{V})\times\mathrm{Mod}(\mathbb{V})^{\mathrm{op}} \to \mathrm{Mod}(\mathbb{V}^{\otimes 2}) \qquad (\mathbb{X},\mathbb{Y}^{\mathrm{op}}) \mapsto \mathbb{X}\otimes_{\mathbb{C}}\mathbb{Y}^\vee$$

- In particular, when $\mathbb{V}$ is rational, we have

$$\boxtimes_{\mathfrak{N}}\mathbb{C} \simeq \bigoplus_{\mathbb{X}\in\mathrm{Irr}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$$

What is an end?

Let $\mathscr{D}$ be a category. Let $F : \mathrm{Mod}(\mathbb{V}^{\otimes N}) \times \mathrm{Mod}(\mathbb{V}^{\otimes N}) \to \mathscr{D}$ be a covariant bi-functor. Let $A \in \mathscr{D}$.

Definition

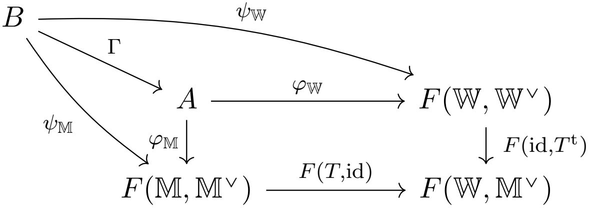
A family of morphisms $\varphi_{\mathbb{W}} : A \to F(\mathbb{W}, \mathbb{W}^\vee)$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called **dinatural** if for any $\mathbb{M} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$ and $T \in \mathrm{Hom}_{\mathbb{V}^{\otimes N}}(\mathbb{M}, \mathbb{W})$, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad \varphi_{\mathbb{W}} \quad} & F(\mathbb{W}, \mathbb{W}^\vee) \\ {\scriptstyle \varphi_{\mathbb{M}}}\big\downarrow & & \big\downarrow{\scriptstyle F(\mathrm{id}, T^{\mathrm{t}})} \\ F(\mathbb{M}, \mathbb{M}^\vee) & \xrightarrow{\quad F(T, \mathrm{id}) \quad} & F(\mathbb{W}, \mathbb{M}^\vee) \end{array}$$

What is an end?

Definition

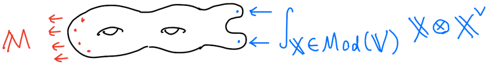
A dinatural transformation $\varphi_{\mathbb{W}} : A \to F(\mathbb{W}, \mathbb{W}^\vee)$ (for all $\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})$) is called an **end** if it satisfies the universal property that for any dinatural transformations $\psi_{\mathbb{W}} : B \to F(\mathbb{W}, \mathbb{W}^\vee)$ there is a unique $\Gamma \in \mathrm{Hom}_{\mathscr{D}}(B, A)$ such that $\psi_{\mathbb{W}} = \varphi_{\mathbb{W}} \circ \Gamma$ for all $\mathbb{W}$. We write $A = \displaystyle\int_{\mathbb{W} \in \mathrm{Mod}(\mathbb{V}^{\otimes N})} F(\mathbb{W}, \mathbb{W}^\vee)$.



The end $\int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X} \otimes \mathbb{X}^\vee$ in the factorization property

Corollary (G.-Zhang, arXiv:2508.04532)

The SF isomorphism defines an $\mathbb{M}$-natural linear isomorphism

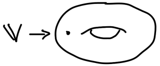
$\mathrm{CB}\Big($ 

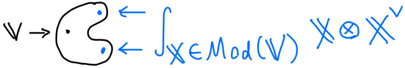
$\Big)$

$\simeq \mathrm{CB}\Big($ 

$\Big)$

- In particular, we have

$\mathrm{CB}\Big($ 

$\Big) \simeq \mathrm{CB}\Big($ 

$\Big)$

- That the factorization property in logarithmic CFT should be understood via ends/coends was proposed by Fuchs-Schweigert (16), following Lyubashenko's approach to non-semisimple TQFT (94).

The end $\int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$ as an associative $\mathbb{C}$-algebra

Theorem (G.-Zhang, arXiv:2508.04532)

The Cardy state space $\mathbb{E} := \displaystyle\int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$ *has a unique structure of an associative* $\mathbb{C}$*-algebra such that the dinatural transform* $\mathbb{E}\to\mathbb{X}\otimes\mathbb{X}^\vee$ *(for all* $\mathbb{X}\in\mathrm{Mod}(\mathbb{V})$*) are homomorphisms of* $\mathbb{C}$*-algebras.*

$\mathbb{E}$ is in general non-unital, since $\mathbb{X}\otimes\mathbb{X}^\vee$ is a non-unital subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{X})$ when $\dim\mathbb{X}=+\infty$. Thus, not all left $\mathbb{E}$-modules are quotients of free modules.

Theorem (G.-Zhang, arXiv:2508.04532)

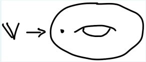
Let $\mathrm{Coh}^{\mathrm{L}}(\mathbb{E})$ *be the category of finitely-generated left* $\mathbb{E}$*-modules that are quotients of free modules. Then the dinatural transform of* $\mathbb{E}$ *defines an equivalence of linear categories* $\mathrm{Coh}^{\mathrm{L}}(\mathbb{E})\simeq\mathrm{Mod}(\mathbb{V})$.

The end $\int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$ as an associative $\mathbb{C}$-algebra

Let $\mathbb{E} := \int_{\mathbb{X}\in\mathrm{Mod}(\mathbb{V})} \mathbb{X}\otimes\mathbb{X}^\vee$. The following result was conjectured by Gainutdinov-Runkel (16). Now, it follows from our Sewing-Factorization theorem.

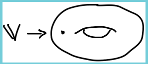
Theorem (G.-Zhang, arXiv:2508.04532)

Let $\mathbb{G}$ be a projective generator of $\mathrm{Mod}(\mathbb{V})$. Then the SF isomorphism defines linear isomorphisms

$$\mathrm{CB}\Big($$


$$\Big) \simeq \mathrm{SLF}(\mathbb{E}) \simeq \mathrm{SLF}(\mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}})$$

- Note that $\mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}}$ is a finite-dimensional unital $\mathbb{C}$-algebra, and $\mathrm{Mod}(\mathbb{V}) \simeq \mathrm{Mod}^{\mathrm{L}}(\mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}})$.

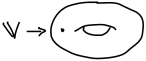
The isomorphism $\mathrm{SLF}(\mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}}) \simeq \mathrm{CB}\Big($  $\Big)$

- Let $\mathbb{G}$ be a projective generator of $\mathrm{Mod}(\mathbb{V})$, graded by generalized eigenspaces of $L(0)$ as $\mathbb{G} = \bigoplus_{\lambda\in\mathbb{C}} \mathbb{G}_{[\lambda]}$.
- Let $A = \mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}}$. Then each $\mathbb{G}_{[\lambda]}$ is a projective right A-module. So there exist

$$\alpha_1, \dots, \alpha_n \in \mathrm{Hom}_A(A, \mathbb{G}_{[\lambda]}) \qquad \breve{\alpha}^1, \dots, \breve{\alpha}^n \in \mathrm{Hom}_A(\mathbb{G}_{[\lambda]}, A)$$

  satisfying $\sum_i \alpha_i \circ \breve{\alpha}^i = \mathrm{id}_{\mathbb{G}_{[\lambda]}}$.
- For each $\omega \in \mathrm{SLF}(A)$, define an SLF $\mathrm{Tr}^\omega_\lambda : \mathrm{End}_A(\mathbb{G}_{[\lambda]}) \to \mathbb{C}$ by

$$\mathrm{Tr}^\omega_\lambda(x) = \sum_i \omega(\breve{\alpha}^i \circ x \circ \alpha_i(1_A))$$

- The linear isomorphism $\mathrm{SLF}(\mathrm{End}_{\mathbb{V}}(\mathbb{G})^{\mathrm{op}}) \xrightarrow{\simeq} \mathrm{CB}\Big($  $\Big)$ sends each ω to the **pseudo-q-trace** (Arike-Nagatomo 11)

$$\mathbb{V} \to \mathbb{C} \qquad v \mapsto \sum_{\lambda\in\mathbb{C}} \mathrm{Tr}^\omega_\lambda\Big(Y_{\mathbb{G}}(v,z) q^{L(0)}\Big|_{\mathbb{G}_{[\lambda]}}\Big)$$