

# Graph Theory

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## Assignment 7

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

**Problem 1:** Determine all positive integers  $r$  and  $s$ , with  $r \leq s$ , for which  $K_{r,s}$  is planar.

**Problem 2:**

- (a) Show that every planar graph has a vertex of degree at most 5. Is there a planar graph with minimum degree 5?
- (b) Show that any planar *bipartite* graph has a vertex of degree at most 3. Is there a planar bipartite graph with minimum degree 3?

**Problem 3:** Show that a connected plane graph  $G$  is bipartite iff all its faces have even length.

**Problem 4:** Let  $G$  be a graph on  $n \geq 3$  vertices and  $3n - 6 + k$  edges for some  $k > 0$ . Show that any drawing of  $G$  in the plane contains at least  $k$  crossing pairs of edges.

**Problem 5:** Let  $G$  be a plane graph with triangular faces and suppose the vertices are colored arbitrarily with three colors. Prove that there is an even number of faces that get all three colors.

**Note:** We remark that the outer face of  $G$  should also be a triangle.

**Problem 6:** Let  $S$  be a set of  $n \geq 3$  points in the plane such that any two of them have distance at least 1. Show that there are at most  $3n - 6$  pairs of distance *exactly* 1.