Graph Theory

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Assignment 6

Unless noted otherwise, all graphs considered are simple. The solution of every problem should be no longer than one page.

Problem 1: Let A be a finite set with subsets $A_1, \ldots, A_n$, and let $d_1, \ldots, d_n$ be positive integers. Show that there are disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for all $k \in [n]$ if and only if

$$|\cup_{i \in I} A_i| \geq \sum_{i \in I} d_i$$

for all $I \subseteq [n]$.

Problem 2: Suppose M is a matching in a bipartite graph $G = (A \cup B, E)$. We say that a path $P = a_1 b_1 \cdots a_k b_k$ is an *augmenting path* in G if $b_i a_{i+1} \in M$ for all $i \in [k-1]$ and a_1 and b_k are not covered by M. The name comes from the fact that the size of M can be increased by flipping the edges along P (in other words, taking the symmetric difference of M and P): by deleting the edges $b_i a_{i+1}$ from M and adding the edges $a_i b_i$ instead.

(a) Prove Hall's theorem by showing that if Hall's condition is satisfied and M does not cover A, then there is an augmenting path in G.

(b) Show that if M is not a maximum matching (i.e. there is a larger matching in G) then the graph contains an augmenting path. Is this true for non-bipartite graphs as well?

Problem 3: Show that for $k \geq 1$, every k-regular $(k-1)$-edge-connected graph on an even number of vertices contains a perfect matching.

Problem 4: An $n \times n$ matrix $A = (a_{i,j})$ is called **doubly stochastic** if all entries are nonnegative and the sum of every row and every column is 1. Show that if A is doubly stochastic then there is a permutation $\sigma : [n] \to [n]$ such that $a_{i,\sigma(i)} > 0$ for every $1 \leq i \leq n$.