Lecture 25 8 June 2022

Potential Theory on infinite graphs

Q: Classification of infinite graphs?

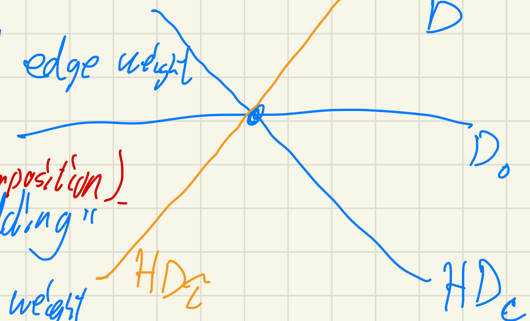
E.g. Recurrent or Transient? O_{HD}? O_{BH}? Strongly Transient (i.e. strong iso. ineq.)

These classification is invariant under some operations.

1). Rough isometry

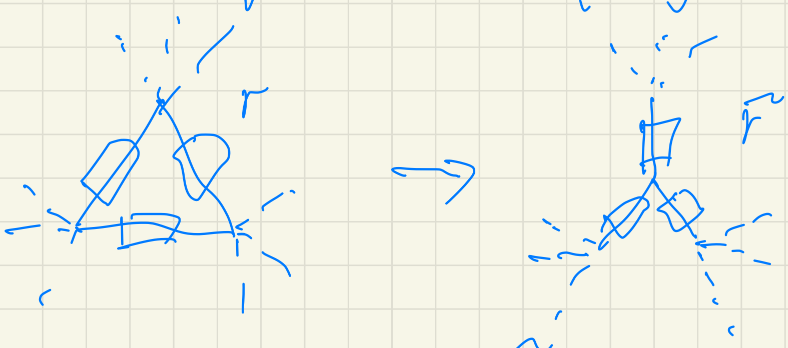
2). Equivalent edge weights: Given $r, \tilde{r}$, equivalent if $\exists K>0$ s.t. $\frac{1}{K}\tilde{r}_{xy} \le r_{xy} \le K\tilde{r}_{xy}$

⇝ geometric edge weight vs combinatorial edge weight
($r = 1$)
(Delaunay decomposition)



Rmk: geometric edge weight from "nice embedding" to metric space is equivalent to comb. edge weight

3). Star-Triangle relation (Yang-Baxter



D $\quad \tilde{D} \quad \rightsquigarrow \quad \dim \tilde{D} = \dim D + 1$

D_0 $\quad \tilde{D}_0 \quad \Rightarrow \quad \dim \tilde{D}_0 = \dim D_0 + 1$

HD $\quad \widetilde{HD} \quad \Rightarrow \quad \dim \widetilde{HD} = \dim HD$.

Tools: Extremal lengths, effective resistance, compactification, ...

CONTENTS

Other related topics (Mostly for finite graphs).

① Spectral graph theory: Eigenvalues of Laplacian $(Id - P)$

② Spanning tree model in statistical mechanics.

Kirchhoff's Thm

~~# {spanning trees}~~ $= \det_0 (C(Id - P))$

weighted sum of spanning trees

$=$ product of non-zero eigen-values.

Normalized Laplacian $= Id - P$.

$\rightsquigarrow \frac{1}{c(x)} \sum_{y \sim x} c_{xy}(f_y - f_x)$

Another version $= C \cdot (Id - P)$

$\rightsquigarrow \sum_{y \sim x} c_{xy}(f_y - f_x)$

$$C = \begin{array}{cc} & x \\ x & \begin{pmatrix} & \vdots & \\ \cdots 0 \cdots & c(x) & 0 \; 0 \; 0 \\ & 0 & \\ & 0 & \\ & 0 & \end{pmatrix} \end{array}$$

③. Geometry

~ Want to find nice realization of Γ into some metric space.

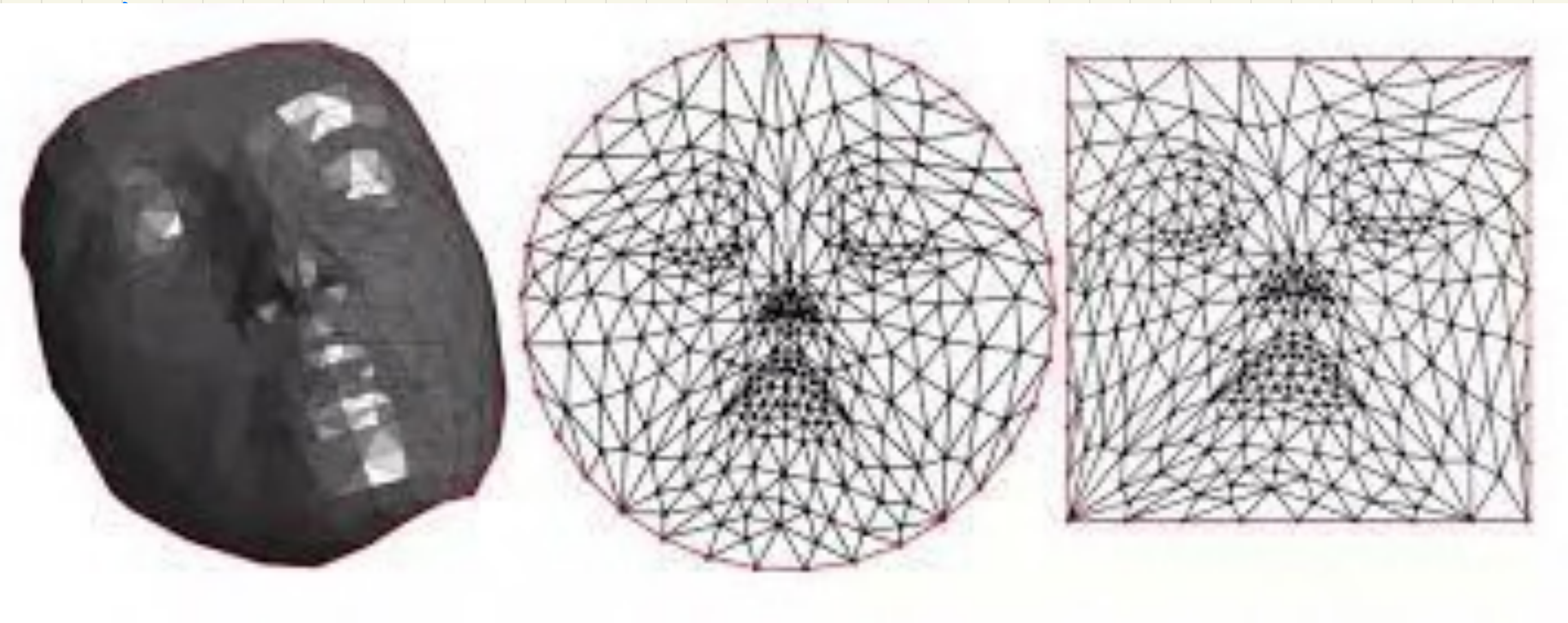
a). Tutte embedding :

Given (Γ, r), find $f : V \to \mathbb{R}^n$

*) minimize Dirichlet energy $\displaystyle\sum_{xy \in E} \frac{1}{r_{xy}} \left(f_x - f_y\right)^2$

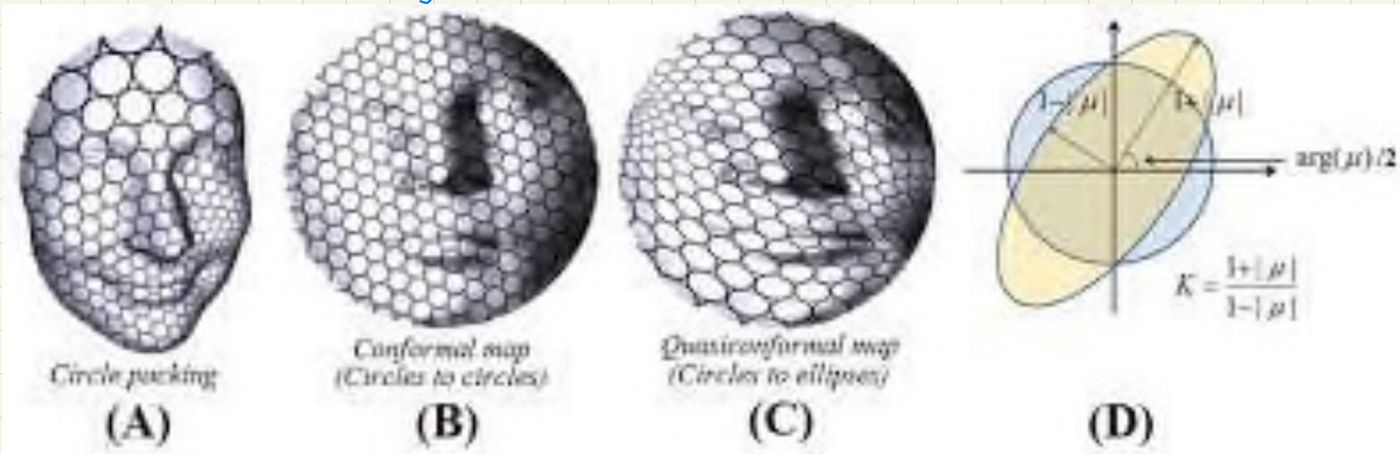
**) f satisfies some boundary condition.

discrete Riemann mapping (linear)



b), Circle packing

discrete Riemann mapping (non-linear)



Fact: Infinitesimal change of radii is a harmonic function.

Weierstrass representation

discrete minimal surface

circle packing

Teichmüller space.

Extra tool for planar graphs: Conjugate harmonic fun.

$\rightsquigarrow$ har. function + Conjugate harm.

$\rightsquigarrow$ discrete holo. function

Q: Extend the theories on finite graphs to infinite graphs?
