

Lecture 25

8 June 2022

Potential Theory on infinite graphs

Q: Classification of infinite graphs?

E.g. Recurrent or Transient? O_{HD} ? O_{BH} ? Strongly Transient
(i.e. strong iso. ineq.)

These classification is invariant under some operations.

- 1). Rough isometry
- 2). Equivalent edge weights: Given $r, \tilde{r}$, equivalent if $\exists K > 0$
s.t. $\frac{1}{K} \tilde{r}_{xy} \leq r_{xy} \leq K \tilde{r}_{xy}$

$\Rightarrow$ geometric edge weight vs combinatorial edge weight

($r=1$)

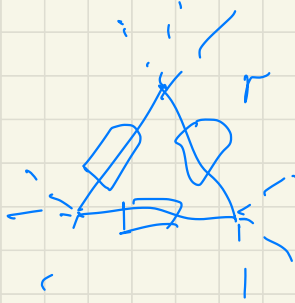
(Delaunay decomposition)

Rank: geometric edge weight from "nice embedding" to metric space is equivalent to comb. edge weight

HD_c

HD_c

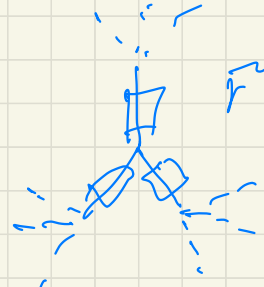
3), Star-Triangle relation / Yang-Baxter



D

D_0

HD



$\tilde{D}$

$\tilde{D}_0$

$\tilde{HD}$

$\Rightarrow$

$$\dim \tilde{D} = \dim D + 1$$

$\Rightarrow$

$$\dim \tilde{D}_0 = \dim D_0 + 1$$

$\Rightarrow$

$$\dim \tilde{HD} = \dim HD$$

Tools: Extremal lengths, effective resistance, compactification, ...

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*Raydon's decomposition
for transient networks
 $D = D_0 \oplus HD$*

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Other related topics (Mostly for finite graphs),

(1). Spectral graph theory: Eigenvalues of Laplacian $(Id - P)$

(2) Spanning tree model in statistical mechanics,

Kerchoff's Thm $\# \{ \text{spanning trees} \} = \det_0 (C (Id - P))$
Weighted sum of spanning trees $=$ product of non-zero eigen-values.

Normalized Laplacian $= Id - P$

$$\Rightarrow \frac{1}{C(x)} \sum_{y \sim x} C_{xy} (f_y - f_x)$$

Another version $= C \cdot (Id - P)$

$$\Rightarrow \sum_{y \sim x} C_{xy} (f_y - f_x)$$

(3). Geometry

~ Want to find nice realization
of Γ into some metric space.

$$C =_x \begin{pmatrix} x \\ \vdots \\ x \\ \vdots \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

a). Tutte embedding:

Given (Γ, r) , find $f: V \rightarrow \mathbb{R}^n$

*) minimize Dirichlet energy $\sum_{x,y \in E} \frac{1}{r_{xy}} |f_x - f_y|^2$

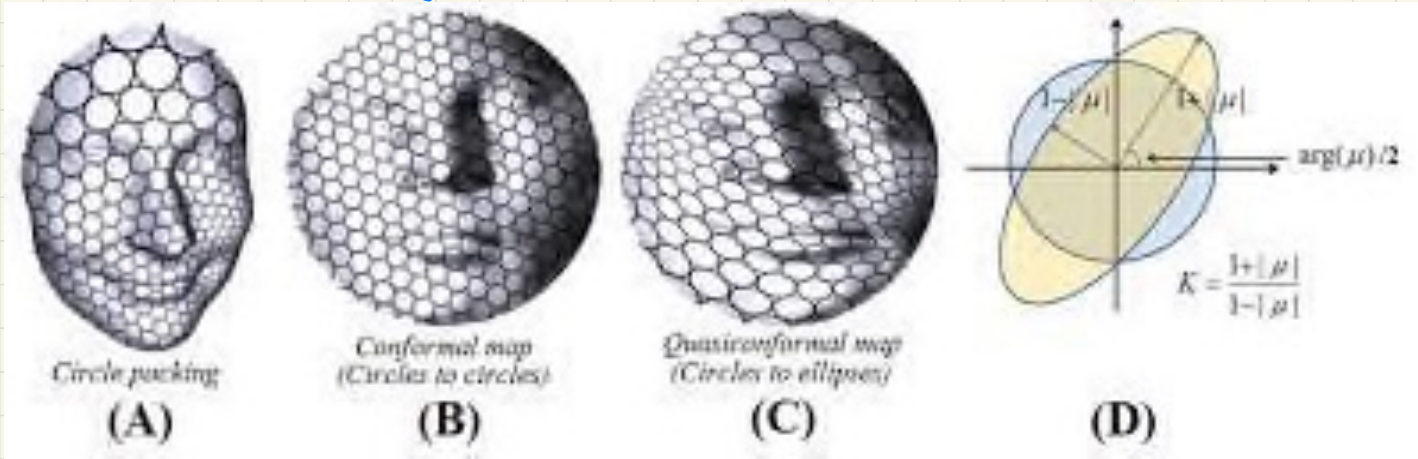
**) f satisfies some boundary condition.

discrete Riemann mapping (linear)



b), Circle packing

discrete Riemann
mapping (non-linear)



Fact: Infinitesimal change of radii is a harmonic function.

Circle packing

Weierstrass representation

discrete minimal surface

Teichmüller space.

Extra tool for planar graphs: Conjugate harmonic Sum.

$\rightarrow$ har. function + Conjugate harm.

$\rightarrow$ discrete holo. function

Q: Extend the theories on finite graphs
to infinite graphs?
