Lecture 24   6 June 2022

Γ does not satisfy strong iso ineq. if for any $K>0$, $\exists$ finite subset L s.t.

$$\frac{|\partial(L)|}{|L|} \leq K$$

Def A locally finite graph has moderate growth if $\exists$ sequence of finite subsets $V_n \subset V$ s.t. $V_n \subseteq V_{n+1}$, $\bigcup_{n=1}^{\infty} V_n = V$

and $\lim_{n\to\infty} \frac{|\partial(V_n)|}{|V_n|} = 0$

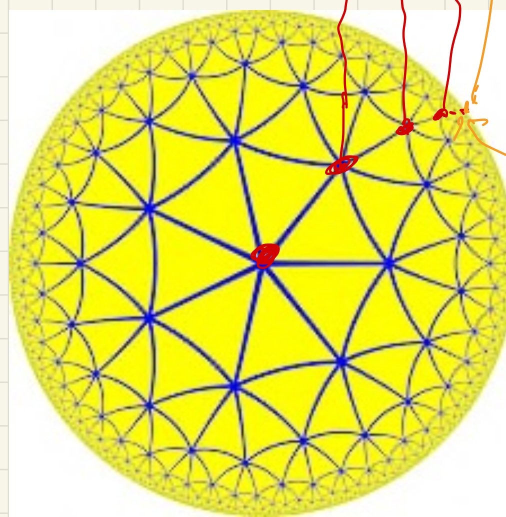
Rmk: moderate growth $\Rightarrow$ Not strong iso. ineq.

Ex: $\mathbb{Z}^2, \mathbb{Z}^n$ has moderate growth

Q: There exists graphs not of moderate growth and fail strong iso inequality?

Possible Example:

Fail strong iso ineq. $\rightarrow$



L_n

$n.$

$|\partial L_n| = 1$

$|L_n| = n$

$$\frac{|\partial L_n|}{|L_n|} = \frac{1}{n}$$

Foster's averaging formula

$$\text{"} \sum_{xy \in E} \frac{R_{eff}(x,y)}{r(x,y)} = |V| - 1 \text{"}$$

It has extension to infinite graphs of moderate growth.

<u>Thm</u> Γ_1, Γ_2 roughly isometric of bounded degree.

Then if Γ_1 satisfies strong iso. ineq.,

then so does Γ_2.

Idea: $\phi: \Gamma_1 \to \Gamma_2$ rough iso. $\exists k$ s.t. $\phi: \Gamma_1 \to \Gamma_2^k$ morphism

Γ_1 strong iso ineq. $\overset{(B)}{\Rightarrow}$ $\Gamma' = \phi(\Gamma_1) \subset \Gamma_2^k$ strong iso ineq.

(C) $\Rightarrow$ Γ_2^K strong iso ineq.

(A) $\Rightarrow$ Γ_2 strong iso ineq.

Lemma (A) Let Γ of bounded degree and L is finite subset of vertices.

$$\partial(L) \subset \Gamma$$

$$\partial^K(L) \subset \Gamma^K$$

Then $\exists K_1 > 0$ independent of L s.t.

$$|\partial(L)| \leq |\partial^K(L)| \leq K_1 |\partial(L)|.$$

[Recall: $x \in \partial(L) \Leftrightarrow x \in L$ and $\exists y \in V$ s.t. $y \notin L$ and $xy \in E(\Gamma)$.

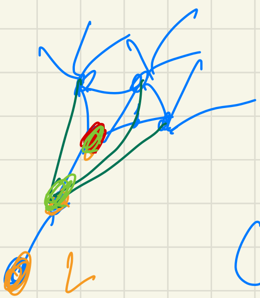
Pf: ($\Rightarrow$) Suppose $x \in T(L)$. Then $\exists\ y \in V-L$ and $xy \in E(\Gamma)$

$$xy \in E(\Gamma) \Rightarrow xy \in E(\Gamma^k)$$

$$\Rightarrow x \in T^k(L)$$

$$\Rightarrow T(L) \subseteq T^k(L)$$

Eg,



$T(L)$

$T^2(L)$

($\Leftarrow$), $T^k(L)$ consists of all vertices in L that has distance d_Γ from $T(L)$ less than $k-1$.

Denote $M = \sup_{x \in V} \deg(x) < \infty$,

$$|T^k(L)| \le |T(L)| \underbrace{(1+M+M^2+M^3+\dots+M^{k-1})}_{K_1}$$

Take $K_1 := \sum_{j=0}^{k-1} M^j$. Then

$$|T^k(L)| \le K_1 |T(L)|$$

Corollary: Γ satisfies strong iso ineq $\Leftrightarrow$ Γ^k ....

Pf: ($\Rightarrow$) Suppose $\exists$ $K \ge 0$ s.t. $\forall$ finite subset L,

$$|T^k(L)| \ge |T(L)| \ge K|L|$$

$\Rightarrow$ Γ^k satisfies strong iso. ineq.

($\Leftarrow$) exercise.

Note: Take K big enough s.t. $\Phi: \Gamma_1 \to \Gamma_2^K$ morphism

<u>Lemma (B)</u> Let $\Gamma' = \Phi(\Gamma_1) \subset \Gamma_2^K$.

Then if Γ_1 satisfies strong iso. ineq. $\Rightarrow$ so does Γ'.

Pf: Consider $f': V' \to \mathbb{R}$ has finite support over Γ'.

Denote $f := f' \circ \Phi$ which has finite support.

i.e. $\forall x \in V$, $f(x) = f'(\Phi(x))$.

Let $M_2 := \sup_{x \in V'} \deg(x)$

$\Phi: \Gamma \to \Gamma'$ surjective.

$$\sum_{x \in V'} \deg(x) |f'(x)|^2 \le M_2 \sum_{x \in V'} |f'(x)|^2 \le M_2 \sum_{x \in V} |f(x)|^2$$

$$\leq M_2 \sum_{x \in V} \deg(x)\, |f(x)|^2$$

$$(\Gamma_1 \text{ satisfies strong iso ineq.}) \leq M_2\, \gamma\, D_{\Gamma_1}(f)$$

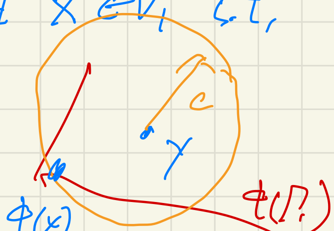
$$\leq M_2\, \gamma\, C\, D_{\Gamma'}(f')$$

$\Rightarrow$ Γ' satisfies strong iso ineq.

Recall $\phi: \Gamma_1 \to \Gamma_2$ rough isometry

$\Rightarrow$ $\exists\ c > 0$ s.t. for any $y \in V_2$, $\exists\ x \in V_1$ s.t.

$$d_{\Gamma_2}(y, \phi(x)) < c$$



Take $K > C$.

$\Rightarrow$ For any $y \in V_2$, $\exists\ x \in V_1$ s.t. $\phi(x)y \in E(\Gamma_2^K)$,

$\Leftrightarrow$ $d_{\Gamma_2^K}(y, \phi(x)) \leq 1$.

<u>Lemma (C)</u> $\Gamma' \subset \Gamma_2^K$ strong iso ineq. $\Rightarrow$ Γ_2^K strong iso ineq,

<u>Pf.</u> Let $f: V_2 \to \mathbb{R}$ function over Γ_2^K with finite support.

Denote $g = f|_{V'}$ function over Γ' with finite support.

$$\sum_{x\in V_2} c(x)\,|f(x)|^2 = \underbrace{\sum_{x\in V'} c(x)\,|f(x)|^2}_{\gamma\, D_{\Gamma'}(g)} + \sum_{x\in V_2 - V'} c(x)\,|f(x)|^2$$

Let $x \in V_2 - V'$, know exists $y \in V'$ s.t. $xy \in E(\Gamma_2^k)$,

$$|f(x)| = |1\cdot(f(x)-f(y)) + 1\cdot(f(y))|$$

$$\le \sqrt{2}\sqrt{|f(x)\cdot f(y)|^2 + |f(y)|^2}$$

$$\Rightarrow \quad |f(x)|^2 \le 2\left(|f(y)|^2 + |f(x)-f(y)|^2\right)$$

$$M_2^k := \sup_{x\in V_2^k} \deg(x)$$

$$\sum_{x\in V_2 - V'} c(x)\,|f(x)|^2 \le M_2^k \sum_{x\in V_2 - V'} |f(x)|^2$$

$$\le M_2^k\left(2M_2^k \sum_{y\in V'} |f(y)|^2 + 2\, D_{\Gamma_2^k}(f)\right)$$

$$\le \hat{M}\left(\sum_{y\in V'} \deg(x)\,|f(y)|^2\right) + 2M_2^k\, D(f)$$

$$\le \gamma \hat{M}\, D_{\Gamma'}(f) + 2M_2^k\, D_{\Gamma_2^k}(f)$$

$$\le \left(\gamma\hat{M} + 2M_2^k\right) D_{\Gamma_2^k}(f)$$

Conclusion: $\exists\, \hat{\gamma} > 0$ s.t. independent of f.

$$\sum_{x\in V_2} c(x)\,|f(x)|^2 \le \hat{\gamma}\, D_{\Gamma_2^k}(f).$$

$\Rightarrow \quad \Gamma_2^K$ satisfies strong iso. ineq.

Thm Let Γ_1, Γ_2 roughly isometric of bounded vertex degree. Then if Γ_2 has moderate growth, Γ_2 also has moderate growth.

Idea: Γ_2 moderate growth $\overset{(A)}{\Longrightarrow}$ Γ_2^K moderate growth

$\Rightarrow \quad \Gamma' = \phi(\Gamma_1) \subset \Gamma_2^K$ moderate growth

$\Rightarrow \quad \Gamma_1$ has moderate growth.

The direction is reversed since

moderate growth is "negation" of strong iso ineq.