Lecture 23 1 - June 2022

Example: Suppose Γ plane graph (with uniformly bounded degree) s.t. Γ is uniformly embeded in $\mathbb{R}^2$, i.e. $\exists\ K > 0$

$$\frac{1}{K} \|z_1 - z_2\|_{\mathbb{R}^2} \le d_\Gamma(z_1, z_2) \le K \|z_1 - z_2\|_{\mathbb{R}^2}$$

Then Γ roughly isometric $\mathbb{R}^2 \sim \mathbb{Z}^2$

$\Rightarrow$ Γ recurrent.

Thm Γ_1, Γ_2 rouhly isometric with bounded vertex degree.

Then $\Gamma_1 \in O_{HD} \Leftrightarrow \Gamma_2 \in O_{HD}$.

Pf: Take $\phi: \Gamma_1 \to \Gamma_2$ rough isometry,

$\exists\, k > 0$ s.t. $\phi: \Gamma_1 \to \Gamma_2^k$ morphism.

$$\Gamma_1 \in \mathcal{O}_{HD} \Rightarrow \phi(\Gamma_1) \underset{\Gamma_2^k}{\in} \mathcal{O}_{HD}$$

True because of the following proposition.

$$\overset{?}{\Rightarrow} \quad \Gamma_2^k \in \mathcal{O}_{HD}$$

$$\Rightarrow \quad \Gamma_2 \in \mathcal{O}_{HD}$$

Prop Let $\Gamma = (V, E)$ transient subgraph of $\Lambda = (L, \mathcal{E})$

s.t. $\Gamma \hookrightarrow \Lambda$ is rough isometry,

Then if $\Gamma \in \mathcal{O}_{HD} \Rightarrow \Lambda \in \mathcal{O}_{HD}$.

Pf Let $f \in HD(\Lambda)$.

Royden decomposition for $D(\Gamma)$: $\underset{\in D(\Gamma)}{f|_\Gamma} = \underset{\in D_0(\Gamma)}{g} + \underset{\in HD(\Gamma) = \{\text{const.}\}.}{k}$

Want to show: $f \equiv k$.

Pick $o \in V$

Denote: P_o one-sided infinite paths in Λ starting from o,

P_o^{Γ} .... infinite paths in Γ ....

Know: For almost all $p \in P_o$, $f(p) := \lim_{\substack{x_n \in p \\ n \to \infty}} f(x_n)$ exists,

(*) (Want to show, for almost all $p \in P_o$, $f(p) = K$.)

In other words: $P^{I} = \{ p \in P_o \mid f(p) = K \}$,

$P^{II} = \{ p \in P_o \mid f(p) \neq K \}$,

$P^{IV} = \{ p \in P_o \mid f(p) \text{ limit not exists} \}$,

$$P_0 = P^{I} \cup P^{II} \cup P^{IV}$$

Know: $\quad \lambda_\Lambda(P^{IV}) = \infty$

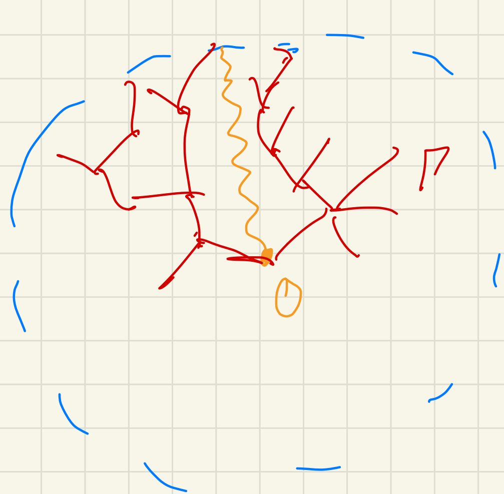
$(*) \iff \lambda_\Lambda(P^{II} \cup P^{IV}) = \infty$

$\iff \lambda_\Lambda(P^{II}) = \infty$

Know: For almost all path in P_0^Γ, $\zeta(p) = K$

$\iff \lambda_\Gamma(P^{I} \cap P_0^\Gamma) = \infty$,

Claim: $\lambda_\Gamma(P^{II} \cap P_0^\Gamma) = \infty \Rightarrow \lambda_\Lambda(P^{II}) = \infty$, Sketch:



$p \in \lambda_\Lambda(p^{\mathrm{I}})$

$\Gamma \hookrightarrow \Lambda$ rough isometry,

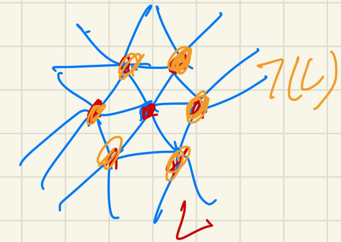
We can project path p to a path $\hat{p}$ on Γ,

s.t. "they are close". $\forall x \in p$, $\exists \hat{x} \in \hat{p}$ s.t.

$$d(x, \hat{x}) < c$$

With this projection, we estimate $L(p)$ in terms of $L(\hat{p})$.

(proofs skipped). $\Rightarrow$ $\lambda_1(P^{\Gamma}) = \infty$.



Strong isoperimetric inequality.

Def L finite subset of vertices in Γ.

Define $\partial(L) \subset L$ s.t. each vertex in $\partial(L)$ has a neighbouring vertex $\notin L$.

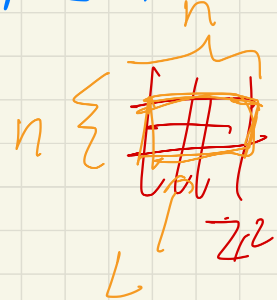
Call $\partial(L)$ combinatorial boundary of L.

Def We say Γ locally finite graph satisfies a strong isoperimetric inequality if $\exists\ \kappa > 0$ st

$$|\partial(L)| \geq \kappa |L| \quad \text{for all finite subset } L.$$

and $|\cdot|$ denotes cardinality.

Example (1) $\mathbb{Z}$, $\mathbb{Z}^2$ fail strong isoperimetric inequality.



No constant $\kappa > 0$ satisfies

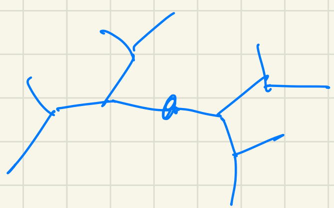
$$4n = \partial(L) > \kappa |L| = n^2 \quad \text{for all } n.$$

(2) Infinite triangulation of disc s.t. $\deg(x) > 7$ for all $x \in V$,

$\Rightarrow$ strong iso. ineq ✓.

(3). Cayley graph of fundamental group of closed surfaces of genus $g > 1$.

$\Rightarrow$ strong iso. ineq ✓.



$$|B_n| = 2^n$$

$$\partial B_n = B_n - B_{n-1} = 2^{n-1}$$

$$|\partial B_n| \text{ ``}=\text{'' } \frac{1}{2} |B_n|$$

$\Rightarrow$ binary tree satisfy strong iso. ineq.

<u>Exercise</u> If Γ satisfies strong inequality, then

$\forall o \in V$, $\quad |B_n(o)| \geq (1+\kappa)^n$

where $B_n(o) := \{ x \in V \mid d(x, o) \leq n \}$,

$\Rightarrow$ number of vertices grow exponentially along distance.

Sketch: $\quad B_{n+1} = B_n + (B_{n+1} - B_n)$

$$|B_{n+1}| \geq |B_n| + |\Gamma B_{n+1}|$$

$$\geq |B_n| + K\,|B_{n+1}|$$

$$\geq |B_n|\,(1+K)$$

$$\Rightarrow \quad |B_n| \geq (1+K)^n\,|B_0| = (1+K)^n$$

Thm Assume Γ (all $c_{xy} \equiv 1$) has uniform bounded vertex degree, i.e. $M := \sup_{x \in V} \deg(x) < \infty$.

Then the following are equiv.

(1). Γ satisfies strong iso. ineq.

(2). $\exists\ \gamma > 0$ s.t. for every function $f: V \to \mathbb{R}$ with finite support, we have

$$\|f\|_2^2 := \sum_{x \in V} c(x) |f(x)|^2 \leq \gamma D(f).$$

where $c(x) = \deg(x)$.

(Poincaré ineq. , Sobolev-Dirichlet ineq).

(3) P has operator norm $\|P\| < 1$.

(4), $\exists\ \sigma$ where $0 < \sigma < 1$ satisfies for some $c > 0$

$$P^n(x,y) = (P^n)_{(x,y)} < C\sigma^n \quad \forall\, x,y \in V.$$

(5) $G := Id + P + P^2 + P^3 + \cdots\cdots = (Id - P)^{-1}$ is

a bounded operator.

$$\left(\text{Recall: } G(x,y) := \lim_{n\to\infty} \sum_{k=0}^{n} P^{(k)}(x,y) \right)$$

Message: (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) proved from definitions.

Pf: (2) $\Rightarrow$ (1).

Take L finite subset.

$$\chi_L(x) := \begin{cases} 1 & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma D(\chi_L) \geq \|\chi_L\|_2^2 = \sum_{x \in L} \deg(x) \cdot 1 \geq |L|,$$

$$\begin{array}{c} \| \\ \gamma \displaystyle\sum_{\substack{x \in \partial(L) \\ y \sim x \\ \text{but } y \notin L}} |\chi_L(x) - \chi_L(y)|^2 \end{array} \leq \gamma M |\partial(L)|$$

$\Rightarrow$ $|\partial(L)| \geq \frac{1}{\gamma M} |L|$ $\Rightarrow$ strong iso, equality $K = \frac{1}{\gamma M}$.

(1) $\Rightarrow$ (2).

Let f any function with finite support.

It take values

$$\cancel{0 <} a_1 < a_2 < \cdots < a_k < \infty.$$

$$L_i := \{ x \in V \mid f(x_i) \geq a_i \}.$$

c!.

$$L_k \subset L_{k-1} \subset \cdots \subset L_1 = \mathrm{supp}(f).$$

$$\Rightarrow \quad f = \sum a_i \, e_{L_i - L_{i+1}}$$

$$= \sum c_i \, e_{L_i}$$ where $c_1 = a_1$, $c_i = a_i - a_{i-1}$

Idea: strong iso. ineq. $\Rightarrow$ $|\partial (L_i)| > \kappa (L_i)$

$\Rightarrow$ $D(e_{L_i}) > \tilde{\kappa} \, \|e_{L_i}\|_2^2$

(Dex.) $\Rightarrow$ $D(f) > \gamma \, \|f\|_2^2$

Remark: strong iso ineq. $\Rightarrow$ ~~strongly~~ transient

because H satisfies (4)
