G connected reductive linear alg. group split/k

The set of parabolic subgroups containing a fixed Borel group is isomorphic to subsets of simple roots. For any $J \subseteq \{\text{simple roots}\}$, we have a decomposition

$$\mathbb{Z}(G/P_J) = \bigoplus_{\bar{w} \in W/W_J} \mathbb{Z}(d_{G/P_J} - l(\bar{w}))[2d_{G/P_J} - 2l(\bar{w})]$$

where $l(\bar{w}) = \min\{l(w) \mid \bar{w} = wW_J\}$. The identification of the index follows from that $W/W_J \cong (G/P_J)^T \subseteq (G/P_J)^{G_m}$ and the diagram

$$\begin{tikzcd} (G/B)^T \arrow[r] \arrow[d, "\wr"] & (G/P_J)^T \arrow[d] \\ (G/B)^{G_m} \arrow[r] & (G/P_J)^{G_m} \end{tikzcd}$$

implies that $(G/P_J)^T = (G/P_J)^{G_m}$.

For example, when $G = GL_n$ and $P = \begin{pmatrix} d\times d & * \\ 0 & (n-d)\times(n-d) \end{pmatrix}$, $G/P = Gr(d,n)$.

The BB decomposition is given by

$$\mathbb{Z}(Gr(d,n)) \cong \bigoplus_{\Gamma} \mathbb{Z}(|\Gamma|)[2|\Gamma|]$$

# of boxes in Γ

where Γ is a Young tableau of size $d \times (n-d)$. Denote by $C_i = C_i(U^\perp)$, where $U^\perp$ is the complement tautological bundle ($rk = n-d$). Then for every Γ = [length $= a_1$, length $= a_2$, …, length $= a_d$], $a_1 \geq a_2 \geq \cdots \geq a_d \geq 0$, $|\Gamma| = \sum_{i=1}^{d} a_i$,

$$C_\Gamma = \begin{vmatrix} C_{a_1} & C_{a_1+1} & \cdots \\ C_{a_2-1} & C_{a_2} & \cdots \\ C_{a_3-2} & C_{a_3-1} & \cdots \\ & & d\times d \end{vmatrix}$$

Giambelli's formula

§10 Category of stabilized motives

In topology, we have $H^{i+1}(X \wedge S^1, \mathbb{Z}) = H^i(X, \mathbb{Z})$ for any pointed space X. Suppose $K(\mathbb{Z}, i)$ is the Eilenberg-MacLane space of $H^i(-,\mathbb{Z})$, i.e. we have

$$[X, K(\mathbb{Z},i)] = H^i(X, \mathbb{Z}).$$

Such space is characterized by $\pi_n(K(\mathbb{Z},i)) = \begin{cases} 0 & n \neq i \\ \mathbb{Z} & n = i \end{cases}$

Then $[X, Map(S^1, K(\mathbb{Z}, i+1))] = [S^1 \wedge X, K(\mathbb{Z}, i+1)] = H^{i+1}(X \wedge S^1, \mathbb{Z}) = H^i(X, \mathbb{Z}) = [X, K(\mathbb{Z}, i)]$.

So we see that $Map(S^1, K(\mathbb{Z}, i+1))$ is weak homotopy equivalent to $K(\mathbb{Z}, i)$. This suggests us to study sequences of spaces $\{E_i\} = E$ with structure maps $S^1 \wedge E_i \to E_{i+1} \Leftrightarrow E_i \to Map(S^1, E_{i+1})$. Such sequences are called spectra.

In our case, by cancellation theorem, suppose K^{n} is Eilenberg-MacLane space of $Spa(n)$-th cohomology of some theory realized in DM. Then we have $\underline{Hom}(G_m^{\wedge}, K^{n+1}) = K^n$ in DM. So it is natural to study G_m-spectra.

But it turns out that it is difficult to define smash product between spectra but easy to define Ω^∞. There is an alternative choice, named symmetric, which is easy to define tensor product but difficult to define Ω^∞. In this course, we mainly introduce the latter construction.

Def 10.1 Let $\mathcal{A}$ be a symmetric closed monoidal abelian category with arbitrary products, i.e. $\mathcal{A}$ admits a tensor product $\otimes$ which is "commutative", "associative", "unitary" and $-\otimes B$ has a right adjoint $\underline{Hom}$. A symmetric sequence of $\mathcal{A}$ is a sequence $(A_n)_{n\in\mathbb{N}}$ s.t. every A_n has an S_n-action. A morphism of symmetric sequences is a collection of S_n-equivariant morphism $f_n: A_n \to B_n$. The category of symmetric sequences is denoted by $\mathcal{A}^S$. (For non-sym case, just forget the S_n-action)

Def 10.2 Suppose $A, B \in \mathcal{A}^S$, we define

$$(A \otimes^S B)_n = \bigoplus_{p=0}^{n} S_n \underset{S_p \times S_{n-p}}{\times} (A_p \otimes B_{n-p})$$

as the tensor product of A and B.

Prop 10.3 The $\mathcal{A}^S$ is a symmetric closed monoidal abelian category.

Proof: The kernel, cokernel are defined termwise so it is abelian. For any $A, B, C \in \mathcal{A}^S$, $(A \otimes^S B) \otimes^S C$ and $A \otimes^S (B \otimes^S C)$ are isomorphic to

$$\bigoplus_{i+j+k=n} S_n \underset{S_i \times S_j \times S_k}{\times} A_i \otimes B_j \otimes C_k$$

so the product is associative.

For any $A, B \in \mathcal{A}^S$, we define $\tau: A \otimes^S B \to B \otimes^S A$ by the following diagram ($i+j=n$)

$$\begin{tikzcd} A_i \otimes B_j \arrow[r] \arrow[d] & B_j \otimes A_i \arrow[r] & S_n \underset{S_j \times S_i}{\times} (B_j \otimes A_i) \arrow[d, "\theta_{j,i}"] \\ S_n \underset{S_i \times S_j}{\times} (A_i \otimes B_j) \arrow[rr, "\tau_{i,j}"] & & S_n \underset{S_j \times S_i}{\times} (B_j \otimes A_i) \end{tikzcd}$$

where $\theta_{ij}: S_n \to S_n$ is the map:

So we can check that it is symmetric monoidal.

To show that it is closed, we define

$$\underline{Hom}^S(A,B)_n = \prod_{p\in\mathbb{N}} \underline{Hom}_{S_p}(A_p, B_{n+p})$$

where $\underline{Hom}_{S_p}(A_p, B_{n+p})$ is the kernel of the map

$$\underline{Hom}(A_p, B_{n+p}) \xrightarrow{(\sigma^* - (1_{S_n}\times\sigma)_*)} \prod_{\sigma\in S_p} \underline{Hom}(A_p, B_{n+p}).$$

Now, giving a morphism from $A \otimes^S B$ to C is equivalent to giving $S_p \times S_q$-equivariant maps

$$f_{p,q}: A_p \otimes B_q \to C_{p+q}.$$

This is equivalent to giving S_p-equivariant maps

$$g_{p,q}: A_p \to \underline{Hom}(B_q, C_{p+q})$$

s.t. for any $\sigma \in S_q$

$$\begin{tikzcd} A_p \arrow[r, "g_{p,q}"] \arrow[d, "g_{p,q}"] & \underline{Hom}(B_q, C_{p+q}) \arrow[d, "\sigma^*"] \\ \underline{Hom}(B_q, C_{p+q}) \arrow[r, "(1_{S_p}\times\sigma)_*"] & \underline{Hom}(B_q, C_{p+q}) \end{tikzcd}$$

This just says that $g_{p,q}$ factors through $\underline{Hom}_{S_p}(B_p, C_{p+q})$. □

We have an adjunction $i_0: \mathcal{A} \rightleftarrows \mathcal{A}^S: ev_0$ where $i_0(A) = (A, 0, 0, \cdots)$ and $ev_0(\{A_n\}) = A_0$.

Def 10.4 For $A \in \mathcal{A}^S$, define $n \in \mathbb{N}$

$$(A\{-n\})_m = \begin{cases} S_m \underset{S_{m-n}}{\times} A_{m-n} & m \geq n \\ 0 & \text{else} \end{cases}$$

$$(A\{n\})_m = Res^{S_{n+m}}_{S_m} A_{n+m}.$$

Then we have an adjunction $\{-i\}: \mathcal{A}^S \rightleftarrows \mathcal{A}^S: \{i\}$, $i \in \mathbb{N}$.

Now suppose $A \in \mathcal{A}$. We define $Sym(A) = (1, A, A\otimes A, A\otimes A\otimes A, \cdots) \in \mathcal{A}^S$, where $(A^{\otimes n}) \circlearrowleft S_n$ is given by permutation of factors.

Prop 10.5 The $Sym(A)$ is a commutative monoid object in $\mathcal{A}^S$.

Proof: There is an obvious unit map $(1, 0, \cdots 0) \to Sym(A)$. We define

$$m: Sym(A) \otimes^S Sym(A) \to Sym(A)$$

by the maps $(A^{\otimes a}) \otimes (A^{\otimes b}) = A^{\otimes a+b}$. Let us prove that the diagram

$$\begin{tikzcd} Sym(A) \otimes^S Sym(A) \arrow[rr, "\tau"] \arrow[dr, "m"'] & & Sym(A) \otimes^S Sym(A) \arrow[dl, "m"] \\ & Sym(A) & \end{tikzcd}$$

commutes. It is equivalent to prove that the diagram

$$\begin{tikzcd} A^{\otimes i} \otimes A^{\otimes j} \arrow[r] & S_n \underset{S_i \times S_j}{\times} (A^{\otimes i} \otimes A^{\otimes j}) \arrow[r, "\tau_{ij}"] \arrow[dr, "m"'] & S_n \underset{S_i \times S_j}{\times} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[d, "m"] \\ & & A^{\otimes n} \end{tikzcd}$$

$n = i+j$

commutes. This follows by the diagram:

$$\begin{tikzcd} A^{\otimes i} \otimes A^{\otimes j} \arrow[r] \arrow[d] & A^{\otimes j} \otimes A^{\otimes i} \arrow[r] \arrow[dr] & S_n \underset{S_j \times S_i}{\times} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[r, "\theta_{j,i}"] \arrow[d] & S_n \underset{S_j \times S_i}{\times} (A^{\otimes j} \otimes A^{\otimes i}) \arrow[d] \\ A^{\otimes n} \arrow[rr, "\theta_{i,j}"] & & A^{\otimes n} \arrow[r, "\theta_{j,i}"] & A^{\otimes n} \end{tikzcd}$$

. □

Def 10.6 Suppose $R \in \mathcal{A}$. Define $Sp_R(\mathcal{A})$ to be the category of $Sym(R)$-mod in $\mathcal{A}^S$, which is called the category of symmetric R-spectra. $Sym(R) \otimes^S E \to E$, associativity, units, $E \in \mathcal{A}^S$.

Equivalently, there are $S_p \times S_q$-equivariant maps $f_{p,q}: R^{\otimes p} \otimes E_q \to E_{p+q}$ s.t. $f_{0,q} = id_{E_q}$ and the diagram

$$\begin{tikzcd} R^{\otimes p} \otimes R^{\otimes q} \otimes E_r \arrow[r, "f_{q,r}"] \arrow[dr, "f_{p+q,r}"'] & R^{\otimes p} \otimes E_{q+r} \arrow[d, "f_{p,q+r}"] \\ & E_{p+q+r} \end{tikzcd}$$

commutes.

This is to say that there maps $R \otimes E_p \to E_{p+1}$ s.t. the composite $R^{\otimes p} \otimes E_q \to R^{\otimes p-1} \otimes E_{q+1} \to \cdots \to E_{p+q}$ is $S_p \times S_q$-equivariant.

Prop 10.7 The $Sp_R(\mathcal{A})$ is a symmetric closed monoidal abelian category.

Proof: Suppose $M, N \in Sp_R(\mathcal{A})$, define $M \otimes N \in Sp_R(\mathcal{A})$ by the exact sequence

$$M \otimes^S Sym(R) \otimes^S N \xrightarrow{m\otimes id_N - id_M \otimes m} M \otimes^S N \to M \otimes N \to 0.$$

Define $\underline{Hom}(M, N) \in Sp_R(\mathcal{A})$ by the exact sequence

$$0 \to \underline{Hom}(M, N) \to \underline{Hom}^S(M, N) \xrightarrow{m^* - m_*} \underline{Hom}^S(Sym(R) \otimes^S M, N)$$

where m_* is given by the composite

$$\underline{Hom}^S(M, N) \xrightarrow{Sym(R)\otimes^S -} \underline{Hom}^S(Sym(R) \otimes^S M, Sym(R) \otimes^S N) \to \underline{Hom}^S(Sym(R) \otimes^S M, N)$$ □

We have an adjunction $Sym(R) \otimes^S -: \mathcal{A}^S \rightleftarrows Sp_R(\mathcal{A}): U$ → forgetful functor. Then we obtain an adjunction

$$(Sym(R) \otimes^S -) \circ i_0 = \Sigma^\infty: \mathcal{A} \rightleftarrows Sp_R(\mathcal{A}): \Omega^\infty = ev_0 \circ U$$

where $\Sigma^\infty A = (A, R\otimes A, R\otimes R\otimes A, \cdots)$ and $\Omega^\infty(E) = E_0$.

infinite suspension

infinite loop space.

We have natural identification $A \otimes^S (B\{-i\}) = (A \otimes^S B)\{-i\}$, $i \in \mathbb{N}$ and a natural map given by the composite

$$A \otimes^S (B\{i\}) \to (A \otimes^S B\{i\})\{-i\}\{i\} = (A \otimes^S B\{i\}\{-i\})\{i\} \to (A \otimes^S B)\{i\}.$$

Restricting the functors $\{-i\}$ and $\{i\}$ to $Sp_R(\mathcal{A})$, we obtain an adjunction

$$\{-i\}: Sp_R(\mathcal{A}) \rightleftarrows Sp_R(\mathcal{A}): \{i\}$$

with $A \otimes (B\{-i\}) = (A \otimes B)\{-i\}$.

Def 10.8 Suppose $S \in Sm/k$. Define $Sp(S) = Sp_{\mathbb{Z}(\hat{G}_m)}(Sh(S))$ and $Sp'(S) = Sp_{\mathbb{Z}(\hat{G}_m)}(PSh(S))$. By termwisely defining, we obtain adjunctions

$$+: Sp'(S) \rightleftarrows Sp(S): U \qquad f^*: Sp(T) \rightleftarrows Sp(S): f_* \quad f: S \to T$$

$$f_\#: Sp(S) \rightleftarrows Sp(T): f^* \quad f: S \to T \text{ smooth.}$$

$$f^*(A \otimes B) = f^*A \otimes f^*B \qquad f_\#(A \otimes_S f^*B) = (f_\# A) \otimes_T B.$$

For any $i \in \mathbb{N}$, $F \in Sh(S)$, we have

$$(\Sigma^\infty F)\{i\} = \Sigma^\infty(\hat{G}_m^{\otimes i} \otimes F).$$

Moreover, for any $X \in Sm/S$, $A, B \in Sp(S)$, we have

$$\underline{Hom}_{Sp(S)}((\Sigma^\infty \mathbb{Z}_S(X))\{-i\}, A) = A_i(X)$$

So every $A \in Sp(S)$ has resolution $L \to A$ of the form of direct sum of $(\Sigma^\infty \mathbb{Z}(X))\{-n\}$.