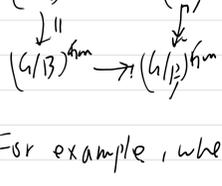


G connected reductive (linear alg. group split/k)
 The set of parabolic subgroups containing a fixed Borel group is isomorphic to subsets of simple roots. For any $J \subseteq \{\text{simple roots}\}$,
 $(J \mapsto P_J)$
 we have a decomposition

$$Z(G/P_J) = \bigoplus_{\bar{w} \in W/J} Z(d_{G/P_J} - l(\bar{w})) [Z(d_{G/P_J} - 2l(\bar{w}))]$$

where $l(\bar{w}) = \min \{ \ell(w) \mid \bar{w} = w \in W/J \}$. The identification of the index follows from that $W/J = [G/P_J]^T \subseteq (G/P_J)^{\text{fm}}$ and the diagram



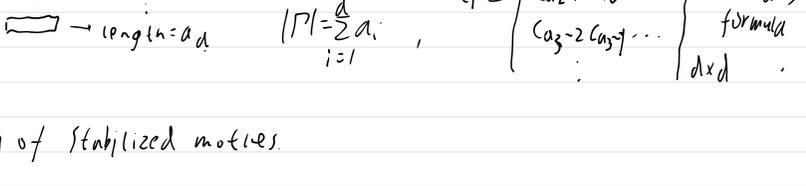
implies that $(G/P_J)^T = (G/P_J)^{\text{fm}}$.

For example, when $G = GL_n$ and $P = \begin{pmatrix} d \times d & * \\ 0 & n-d \times n-d \end{pmatrix}$. $G/P = GL(d, n)$.
 The BB decomposition is given by

$$Z(GL(d, n)) \cong \bigoplus_{\Gamma} Z(\Gamma) [Z(\Gamma)]$$

+ no. of boxes in Γ

where Γ is a Young tableau of size $d \times (n-d)$. Denote by $c_i = c_i(u)$, where u^i is the complement topological bundle ($rk = n-d$). Then for every $\Gamma =$



Sto category of stabilized motives.

In topology, we have $H^{i+1}(X, \mathbb{Z}) = H^i(X, \mathbb{Z})$ for any pointed space X .
 Suppose $k(\mathbb{Z}, 1)$ is the Eilenberg-MacLane space of $H^i(-, \mathbb{Z})$, i.e. we have

$$[X, k(\mathbb{Z}, 1)] = H^i(X, \mathbb{Z})$$

Such space is characterized by $\pi_n(k(\mathbb{Z}, 1)) = \begin{cases} \mathbb{Z} & n=i \\ 0 & n \neq i \end{cases}$.

Then $[X, \text{Map}(S^1, k(\mathbb{Z}, 1))] = [X, S^1, k(\mathbb{Z}, 1)] = \text{Hit}(X, S^1, \mathbb{Z}) = H^i(X, \mathbb{Z}) = [X, k(\mathbb{Z}, 1)]$.

So we see that $\text{Map}(S^1, k(\mathbb{Z}, 1))$ is weak homotopy equivalent to $k(\mathbb{Z}, 1)$. This suggests us to study sequences of spaces $\{E_i\} = E$ with structure maps $S^1 \wedge E_i \rightarrow E_{i+1} \rightarrow E_i \rightarrow \text{Map}(S^1, E_{i+1})$. Such sequences are called spectra.

In our case, by cancellation theorem, suppose $k^{\mathbb{R}} \wedge A$ is Eilenberg-MacLane space of $(\mathbb{R}/\pi) \wedge A$ (homology of some theory realized in DM). Then we have $\text{Hom}(S^1, k^{\mathbb{R}} \wedge A) = k^{\mathbb{R}} \wedge A$ in DM. So it is natural to study \mathbb{R}/π -spectra.

But it turns out that it is difficult to define smash product between spectra but easy to define S^1 . There is an alternative choice, named symmetric, which easy to define tensor product but difficult to define S^1 . In this course, we mainly introduce the latter construction.

Def 10.1 Let \mathcal{A} be a symmetric closed monoidal abelian category with arbitrary products, i.e. \mathcal{A} admits a tensor product \otimes which is "commutative", "associative", "unitary" and \otimes has a right adjoint Hom . A symmetric sequence of \mathcal{A} is a sequence $(A_n)_{n \in \mathbb{N}}$ s.t. every A_n has an S_n -action. A morphism of symmetric sequences is a collection of S_n -equivariant morphism $f_n: A_n \rightarrow B_n$. The category of symmetric sequences is denoted by \mathcal{A}^S . (For non-sym case, just forget the S_n -action)

Def 10.2 Suppose $A, B \in \mathcal{A}^S$, we define

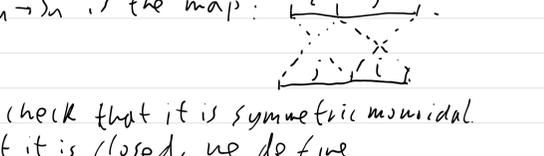
$$(A \otimes^S B)_n = \bigoplus_{p+q=n} S_n \times (A_p \otimes B_q)$$

as the tensor product of A and B .

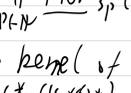
Prop 10.3 The \mathcal{A}^S is a symmetric closed monoidal abelian category.
 Proof: The kernel, cokernel are defined termwise so it is abelian. For any $A, B, C \in \mathcal{A}^S$, $(A \otimes^S B) \otimes^S C$ and $A \otimes^S (B \otimes^S C)$ are isomorphic to

$$\bigoplus_{i+j+k=n} S_n \times (A_i \otimes B_j \otimes C_k) \text{ so the product is associative.}$$

For any $A, B, C \in \mathcal{A}^S$, we define $\tau: A \otimes^S B \rightarrow B \otimes^S A$ by the following diagram ($i+j=n$)



where $\theta_{ij}: S_n \rightarrow S_n$ is the map:



So we can check that it is symmetric monoidal.
 To show that it is closed, we define

$$\text{Hom}^S(A, B)_n = \prod_{p+q=n} \text{Hom}_{S_p} (A_p, B_{q+p})$$

where $\text{Hom}_{S_p}(A_p, B_{q+p})$ is the kernel of the map

$$\text{Hom}(A_p, B_{q+p}) \xrightarrow{\bigoplus_{\sigma \in S_p} (\sigma^* - 1)} \text{Hom}(A_p, B_{q+p})$$

Now, giving a morphism from $A \otimes^S B$ to C is equivalent to giving $S_p \times S_q$ -equivariant maps

$$f_{pq}: A_p \otimes B_q \rightarrow C_{p+q}$$

This is equivalent to giving S_p -equivariant maps

$$g_{pq}: A_p \rightarrow \text{Hom}(B_q, C_{p+q})$$

s.t. for any $\sigma \in S_q$

$$A_p \xrightarrow{g_{pq}} \text{Hom}(B_q, C_{p+q}) \xrightarrow{\sigma^*} \text{Hom}(B_q, C_{p+q})$$

$$\downarrow g_{pq} \quad \downarrow \sigma^* \quad \downarrow \sigma^*$$

$$\text{Hom}(B_q, C_{p+q}) \xrightarrow{\quad} \text{Hom}(B_q, C_{p+q})$$

This just says that g_{pq} factors through $\text{Hom}_{S_p}(B_q, C_{p+q})$. \square

We have an adjunction $i_*: \mathcal{A} \rightleftharpoons \mathcal{A}^S: \nu_*$ where $i_*(A) = (A, 0, \dots)$ and $\nu_*(\varphi_n) = \varphi_n$.

Def 10.4 For $A \in \mathcal{A}^S$, define $(A[-i])_m = \begin{cases} S_m \times A_{m-i} & m \geq i \\ 0 & \text{else} \end{cases}$

$$(A\{i\})_m = \text{Res}_{S_m}^{S_{m+i}} A_{m+i}$$

Then we have an adjunction $\{-i\}: \mathcal{A}^S \rightleftharpoons \mathcal{A}^S: \{i\}, i \in \mathbb{N}$.

Now suppose $A \in \mathcal{A}$. We define $\text{Sym}(A) = (1, A, A \otimes A, A \otimes A \otimes A, \dots) \in \mathcal{A}^S$, where $(A^{\otimes n})_{S_n}$ is given by permutation of factors.

Prop 10.5 The $\text{Sym}(A)$ is a commutative monoidal object in \mathcal{A}^S .

Proof: There is an obvious unit map $(1, 0, \dots) \rightarrow \text{Sym}(A)$. We define

$$m: \text{Sym}(A) \otimes^S \text{Sym}(A) \rightarrow \text{Sym}(A)$$

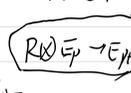
by the maps $(A^{\otimes a}) \otimes (A^{\otimes b}) = A^{\otimes(a+b)}$. Let us prove that the diagram

$$\text{Sym}(A) \otimes^S \text{Sym}(A) \xrightarrow{\quad} \text{Sym}(A) \otimes^S \text{Sym}(A)$$



commutes. It is equivalent to prove that the diagram

$$A^{\otimes i} \otimes A^{\otimes j} \rightarrow S_n \times (A^{\otimes i} \otimes A^{\otimes j}) \xrightarrow{\tau_{ij}} S_n \times (A^{\otimes i} \otimes A^{\otimes j})$$



commutes. This follows by the diagram:

$$\begin{array}{ccccc} A^{\otimes i} \otimes A^{\otimes j} & \rightarrow & A^{\otimes i} \otimes A^{\otimes j} & \rightarrow & S_n \times (A^{\otimes i} \otimes A^{\otimes j}) \xrightarrow{\theta_{ij}} S_n \times (A^{\otimes i} \otimes A^{\otimes j}) \\ & \searrow & \downarrow & \searrow & \downarrow \theta_{ij} \\ & & A^{\otimes n} & & A^{\otimes n} \end{array} \quad D$$

Def 10.6 Suppose $R \in \mathcal{A}$. Define $\text{Sp}_R(\mathcal{A})$ to be the category of $\text{Sym}(R)$ -mod in \mathcal{A}^S , which is called the category of symmetric R -spectra. $\text{Sym}(R) \otimes^S E \rightarrow E$ s.t. $f \circ g = \text{id}_E$ and the diagram

$$\begin{array}{ccc} R^{\otimes p} \otimes R^{\otimes q} \otimes E & \xrightarrow{f \circ g} & E_{p+q} \\ \downarrow & & \downarrow \\ R^{\otimes p} \otimes E_{q+r} & \xrightarrow{f \circ g} & E_{p+r} \end{array}$$

commutes. This is to say that these maps $(R \otimes E_p \rightarrow E_{p+1})$ s.t. the composite $R^{\otimes p} \otimes E_q \rightarrow R^{\otimes p} \otimes E_{q+1} \rightarrow \dots \rightarrow E_{p+q}$ is $S_p \times S_q$ -equivariant.

Prop 10.7 The $\text{Sp}_R(\mathcal{A})$ is a symmetric closed monoidal abelian category.
 Proof: Suppose $M, N \in \text{Sp}_R(\mathcal{A})$, define $M \otimes N \in \text{Sp}_R(\mathcal{A})$ by the exact sequence

$$M \otimes^S \text{Sym}(R) \otimes N \xrightarrow{\text{multiplication}} M \otimes N \rightarrow 0$$

Define $\text{Hom}(M, N) \in \text{Sp}_R(\mathcal{A})$ by the exact sequence

$$0 \rightarrow \text{Hom}(M, N) \rightarrow \text{Hom}^S(M, N) \xrightarrow{m^*} \text{Hom}^S(\text{Sym}(R) \otimes M, N)$$

where m^* is given by the composite

$$\text{Hom}(M, N) \xrightarrow{\text{Sym}(R) \otimes} \text{Hom}^S(\text{Sym}(R) \otimes M, \text{Sym}(R) \otimes N) \rightarrow \text{Hom}^S(\text{Sym}(R) \otimes M, N)$$

We have an adjunction $\text{Sym}(R) \otimes^S \mathcal{A}^S \rightleftharpoons \text{Sp}_R(\mathcal{A}): \nu$ forgetful functor. Then we obtain an adjunction

$$(\text{Sym}(R) \otimes^S -)_i \circ \Sigma^\infty: \mathcal{A} \rightleftharpoons \text{Sp}_R(\mathcal{A}): \Sigma^\infty = \nu_* \circ \nu^*$$

where $\Sigma^\infty A = (A, R \otimes A, R \otimes R \otimes A, \dots)$ and $\Sigma^\infty(E) = E_*$.

infinite suspension infinite loop space.

We have natural identification $A \otimes^S (B[-i]) = (A \otimes^S B)[-i]$ and a natural map given by the composite

$$A \otimes^S (B[-i]) \rightarrow (A \otimes^S B)[-i] \{-i\} \{i\} = (A \otimes^S B[-i]) \{i\} \{i\} \rightarrow (A \otimes^S B) \{i\}.$$

Restricting the functors $\{-i\}$ and $\{i\}$ to $\text{Sp}_R(\mathcal{A})$, we obtain an adjunction

$$\{-i\}: \text{Sp}_R(\mathcal{A}) \rightleftharpoons \text{Sp}_R(\mathcal{A}): \{i\}$$

with $A \otimes B \{-i\} = (A \otimes B)[-i]$.

Def 10.8 Suppose $S \in \text{Sp}_R(\mathcal{A})$. Define $\text{Sp}(S) = \text{Sp}_{\text{Sym}(R)}(\text{Sh}(S))$ and $\text{Sp}'(S) = \text{Sp}_{\text{Sym}(R)}(\text{PSh}(S))$. By termwise defining, we obtain an adjunction

$$+ : \text{Sp}'(S) \rightleftharpoons \text{Sp}(S): \cup \quad f^*: \text{Sp}(T) \rightleftharpoons \text{Sp}(S): f_* \quad f: S \rightarrow T \text{ mon.}$$

$$f^*: \text{Sp}(T) \rightleftharpoons \text{Sp}(S): f_* \quad f: S \rightarrow T \text{ mon.}$$

$$f^*(A \otimes B) = f^*A \otimes f^*B \quad f_*(A \otimes^* B) = (f_*A) \otimes^* B$$

For any $i \in \mathbb{N}$, $F(-) \text{Sh}(S)$, we have

$$(\Sigma^\infty F) \{i\} = \Sigma^\infty (F \wedge i) \otimes F$$

more over, for any $X \in \text{Sp}(S)$, $A, B \in \text{Sp}(S)$, we have

$$\text{Hom}_{\text{Sp}(S)}((\Sigma^\infty \mathbb{Z}_2(X)) \{-i\}, A) = A_i(X)$$

So every $A \in \text{Sp}(S)$ has resolution $L \rightarrow A$ of the form of direct sum of $(\Sigma^\infty \mathbb{Z}_2(X)) \{-n\}$.