Lecture 21 25-May 2022

Rough isometry (Cont.)

Bounded vertex degree : $\exists M > 0$ s.t.

$$\sup_{x \in V} \deg x < M.$$

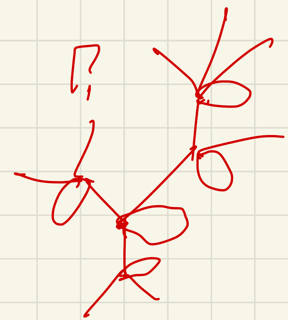
Def Given $\Gamma = (V, Y)$ locally finite graph and $K \in \mathbb{Z}_{>0}$,

We define $\Gamma_K = (V_K, Y_{(K)})$ K-fuzz of Γ as follows.

$$V_K = V$$

$$\Upsilon_K := \{ (x,y) \in V \times V \mid d(x,y) \le K \},$$

Ex:



Observation: (1) $\forall\ x, y \in V$

$$d_{\Gamma_K}(x,y) \le d_{\Gamma}(x,y) \le K\, d_{\Gamma_K}(x,y)$$

$\Rightarrow$ Γ_K, Γ is rough isometric. $\phi : V \to V_K$, $x \mapsto x$

(2) $\phi : \Gamma \to \Gamma$ rough isometric

$\phi : \Gamma \to \Gamma_K$ rough isometric

<u>Prop</u> Suppose Γ has bounded vertex degree, and $K \in \mathbb{Z}_{>0}$.

Then, Γ recurrent $\Leftrightarrow$ Γ_K recurrent

$\Gamma \in O_{HD}$ $\Leftrightarrow$ $\Gamma_K \in O_{HD}$.

<u>Pf</u> Goal: To show the norms $\|\cdot\|_{D(\Gamma)}$, $\|\cdot\|_{D(\Gamma_K)}$ are equivalent.

Pick any $f : V \to \mathbb{R}$, $\quad Y \subset Y_1 \subset Y_2 \subset \ldots$

$$\sum_{xy \in Y} |f_x - f_y|^2 \le \sum_{xy \in Y_k} |f_x - f_y|^2$$

$$\Rightarrow \quad D_\Gamma(f) \le D_{\Gamma_k}(f)$$

$$\Rightarrow \quad \|f\|_\Gamma \le \|f\|_{\Gamma_k}$$

The other direction: For $e = [x_1, x_2] \in Y_k$

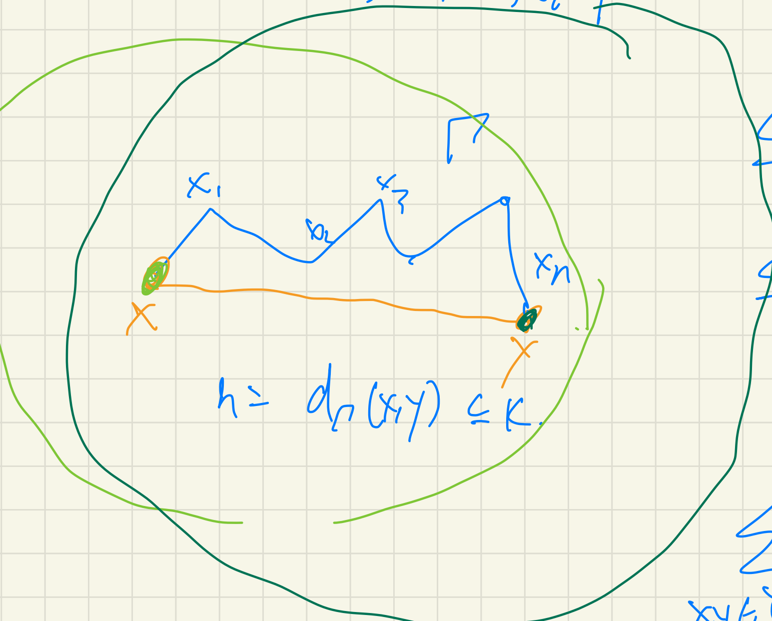
$$U(e, k) = \{ [z_1, z_2] \in Y \mid d(x_i, z_i) \le k \text{ for } i = 1, 2 \}$$

Pick any $x,y \in V$ s.t. $1 \le d_\Gamma(x,y) \le K$ $\left(\Rightarrow xy \in \Upsilon_K\right)$

$$|f(x)-f(y)|^2 \le \left(1\cdot|f(x)-f(x_1)| + 1\cdot|f(x_1)-f(x_2)| + \dots + 1\cdot|\ \ |\right)^2$$

$$\le K \sum |f(x_i)-f(x_{i+1})|^2$$

$$\le K \sum_{st \in V(\{x,y\},K)} |f(s)-f(t)|^2$$



$$\sum_{xy \in \Upsilon_K} |f(x)-f(y)|^2 \le K \overbrace{\sum_{xy \in \Upsilon_K} \sum_{st \in V(\{x,y\},K)}}^{\star} |f(s)-f(t)|^2$$

$$\le K M^{2K} \sum_{xy \in E} |f(s)-f(t)|^2$$

Counting: Fix $x \in V$,

$\#\{$ vertices with distance from $x \leq K\} \leq M^K$

and any two vertices might produce an edge in Γ_K

$\Rightarrow$ $[x,y] \in Y$ appear at most M^{2K} times in $(*)$

$$D_{\Gamma_K}(f) \leq K M^{2K} D_{\Gamma}(f)$$

$$\Rightarrow \quad \|\cdot\|_{\Gamma_K} \leq K M^{2K} \|\cdot\|_{\Gamma}$$

Norms equivalent $\Rightarrow$ $D(\Gamma) = D(\Gamma_K)$

$D_0(\Gamma) = D_0(\Gamma_K)$

Γ recurrent $\Leftrightarrow$

$$\begin{array}{ccc} D(\Gamma) & = & D_0(\Gamma) \\ \shortparallel & & \shortparallel \\ D(\Gamma_K) & & D_0(\Gamma_K) \end{array}$$

$\Leftrightarrow$ Γ_K recurrent,

Assume Γ transient and $\Gamma \in O_{HD}$,

Let $h \in HD(\Gamma_K)$.

$\Rightarrow$ $h \in D(\Gamma)$

Royden decomposition of $D(\Gamma)$: $h = f + g$

where $f \in D_0(\Gamma)$, $g \in HD(\Gamma)$

$\Updownarrow$ since $\Gamma \in O_{HD}$,

g const.

Note: $f \in D_0(\Gamma_K)$, $g \in HD(\Gamma_K)$

$\Rightarrow$ $0 + h = f + g$ Royden decomposition of $D(\Gamma_K)$.

By uniqueness,

$\Rightarrow$

$$f = 0$$

$$h = g \quad \text{const.}$$

$\Rightarrow$ $\Gamma_K \in O_{HD}$

Def Let φ morphism from Γ_1 to Γ_2.

$$\begin{pmatrix} V_1 \to V_2 \\ Y_1 \to Y_2 \end{pmatrix}$$

We define the image of φ is graph

$$\Gamma_1' = (V_1', Y_1')$$

$$V_1' = \phi(V_1), \qquad Y_1' = \phi(Y_1) \subset Y_2.$$

$\Rightarrow \Gamma_1'$ is connected. subgraph of Γ_2.

Prop Let Γ, Γ' graphs of bounded degree.

Suppose ψ morphism from Γ to Γ' s.t.

(1) Γ' image of Γ under ψ,

(2) $\exists\, m > 0$, satisfying for $x, y \in V$

$$\psi(x) = \psi(y) \Rightarrow d_\Gamma(x, y) \leq m.$$

Then, if Γ transient $\Rightarrow$ Γ' transient

Γ O_{HD} $\Rightarrow$ Γ' O_{HD}.

Keep in mind! $\psi : \Gamma \to \Gamma'$

PS For $f': V' \to \mathbb{R}$, consider

$$f(x) := f'(\psi(x)) \quad \forall x \in V$$

$$\Rightarrow f: V \to \mathbb{R}. \quad (f := f' \circ \psi)$$

Goal: $\psi^* D(\Gamma') < D(\Gamma)$

Relate $\|f'\|_{\Gamma'} \sim \|f\|_{\Gamma}$

Claim: $\|f'\|_{\Gamma'} \leq \|f\|_{\Gamma}$ for all $f': V' \to \mathbb{R}$.

Pf

$$\sum_{x'y' \in Y'} (f'(x') - f'(y'))^2 \leq \sum_{xy \in E} (f(x) - f(y))^2$$

$$\Rightarrow \|f'\|_{\Gamma'} \leq \|f\|_{\Gamma}$$

Note: for $x'y' \in Y'$, $\exists\ xy \in Y$ s.t. $\varphi(xy) = x'y'$

$$|f(x) - f(y)|^2 = |f'(x) - f'(y)|^2$$

Claim: $\exists\ C > 0$ s.t. $\|f\|_{\Gamma} \leq C\|f'\|_{\Gamma'}$ $\forall f'$.

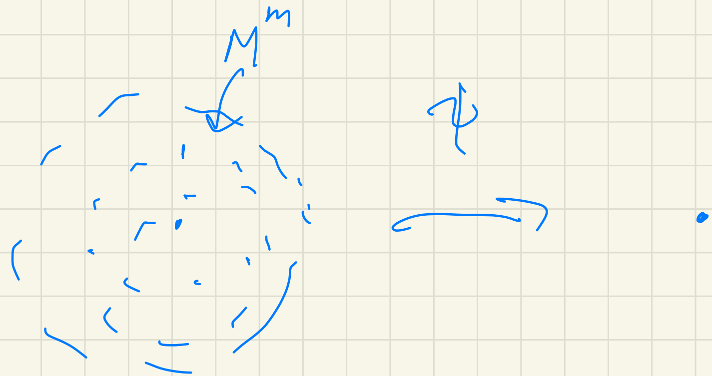
(Exercise)

Suppose Γ' recurrent, and $e' \equiv 1$ const function on Γ'.

$\Rightarrow$ $e' \in D_0(\Gamma')$

$\Rightarrow$ $\exists\ f_n : V' \to \mathbb{R}$ with finite support s.t. $\quad \| e - f_n' \|_{\Gamma''} \to 0 \quad$ as $\quad n \to \infty$

Consider $\quad f_n := f_n' \circ \psi ,$



$\Rightarrow \quad f_n \quad$ has finite support.

$\Rightarrow \quad e := e' \circ \psi \quad$ const. on Γ.

$$\| e - f_n \|_{\Gamma} \le C \| e' - f_n' \|_{\Gamma'} \to 0$$
as $n \to \infty$

$\Rightarrow$ $e \in D_0(\Gamma)$

$\Rightarrow$ Γ is recurrent.

Conclusion

Γ' recurrent $\Rightarrow$ Γ recurrent

Γ transient $\Rightarrow$ Γ' transient.

Question:

~~Remark:~~ Converse might not hold. ?