Lecture 20

23-May 2022

Thm Suppose $f \in HD$ and nonconstant such that

$$f|_{\Delta} \geq 0$$

Then $f > 0$ on V

Thm Assume (Γ, r) transient.

$$D_0 = \{ f \in D \mid f(x) = 0 \quad \forall x \in \Delta \}.$$

Recall: Given one-sided infinite paths p

Extremal points: $E(p) = \overline{V(p)} - V(p) \subset bR$

Thm Assume (Γ, r) transient. And $\underline{P}$ family of one-sided infinite paths in Γ such that

$$F := \overline{\bigcup_{p \in \underline{P}} \Xi(p)} \subset bR$$

is disjoint from Δ. Then $\lambda(\underline{P}) = \infty$

$$\Leftrightarrow \quad \lambda^{-1}(\underline{P}) = 0$$

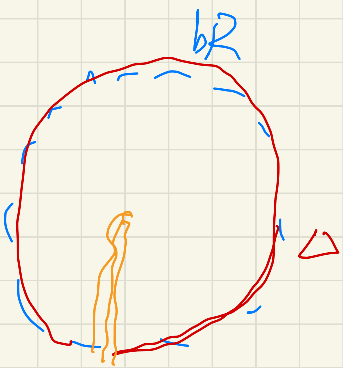
Recall: If a path p satisfies $\sum_n r(x_n, x_{n+1}) < \infty$

then $\Xi(p) \in \Delta$.

Corollary: Suppose $\mathcal{P}$ family of one-sided inf. paths

s.t. $\lambda(\mathcal{P}) < \infty$.

$\Rightarrow \overline{\bigcup_{p \in \mathcal{P}} E(p)} \cap \Delta \neq \phi$.



Rmk: If (Γ, r) is recurrent,

we know already $\Delta = \phi$

and $\lambda(\mathcal{P}) = \infty$.

Ex, $\mathcal{P} = \{p\}$ where p is one-sided inf path

s.t. $\sum_n r(x_n, x_{n+1}) < \infty$

then $\lambda(\mathcal{P}) < \infty$.

Thm (Harmonic measure),

Let (Γ, r) transient. For every $z \in V$, there exists a unique positive regular Borel measure μ_z on bR satisfying the following:

(1), $\mu_z(U) > 0$ $\forall$ all open sets $U \subset bR$ s.t. $U \cap \Delta \neq \phi$.

(2), $\mathrm{Support}(\mu_z) = \Delta$.

(3), every $f \in HD$ is μ_z-integrable and

$$f(z) = \int_{\Delta} f(x)\, d\mu_z(x)$$

(4), $\mu_z(\Delta) = 1$.

Comparison with harmonic functions on unit disk:

$$f(z) = f(re^{i\theta}) = \int_0^{2\pi} f(e^{it}) \, \mathrm{Re}\left(\frac{1+re^{i(\theta-t)}}{1-re^{i(\theta-t)}}\right) dt$$

$z = re^{i\theta}$

$0 \le r \le 1$

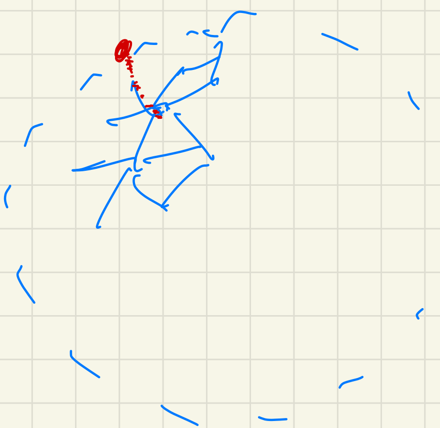
$0 \le \theta < 2\pi$

$$= \int_0^{2\pi} f(e^{it}) \, \underbrace{\mathrm{Re}\left(\frac{1+ze^{-it}}{1-ze^{-it}}\right) dt}_{d\mu_z(t)}$$

Rmk $\mathcal{M}_Z$ depends very much on edge weights.

$\Rightarrow$ boundary value problem depends on edge weights.

Hutchcroft (2019).



$\Rightarrow$ boundary value problem that is not sensitive with edge weights but depend on embeddings.

Rough isometry = Gromov's quasi-isometry.

Def Suppose (X_1, d_1), (X_2, d_2) are metric spaces. A map $\phi: X_1 \to X_2$ is called rough isometry if

(1). $\exists$ $a > 0$, $b \geq 0$ s.t. $\forall$ $x, y \in X_1$

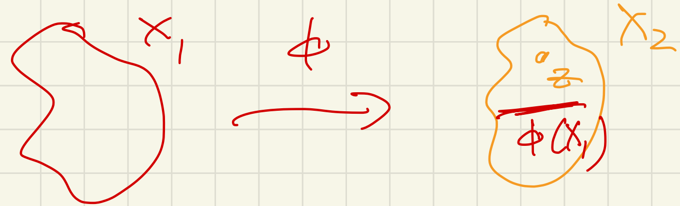
$$a^{-1} d_1(x,y) - b \leq d_2(\phi(x), \phi(y)) \leq a\, d_1(x,y) + b$$

(2). $\exists$ $C > 0$ s.t. for all $z \in X_2$, $\exists$ $x \in X_1$ s.t.

$$d_2(z, \phi(x)) \leq C.$$

(1), $\Leftrightarrow$ d_1, d_2 comparable asymptotically

(2),



We call a, b, c constants of rough isometry

and X_1, X_2 are roughly isometric.

Remark: (1) Generally, ϕ is not surjective

is not continous.

(2) rough isometry is meaningful only if

$$\begin{array}{c} \operatorname{diam}(X_2, d_2) = \infty = \operatorname{diam}(X_1, d_1). \\ \parallel \\ \sup(\, d(x,y) \mid x, y \in X_2 \,) \end{array}$$

Claim: If $\operatorname{diam}(X_2, d_2) < \infty$, $\operatorname{diam}(X_1, d_1) < \infty$,

then for any $\phi : X_1 \to X_2$,

such constants a, b, c always exists,

If $\operatorname{diam}(X_2, d_2) < \infty$ and $\operatorname{diam}(X_1, d_2) = \infty$

then no rough isometry.

Thm Roughly isometric is an equivalence relation on metric spaces.

pf (1) Transitive: $\phi : (X, d_1) \to (Y, d_2)$, $\psi : (Y, d_2) \to (Z, d_3)$ $\Big\}$ rough isometry

Claim: $\Rightarrow$ $\psi \circ \phi : (X, d_1) \to (Z, d_3)$ is rough isometry

$\exists\ a_1, b_1$ s.t. $a_1^{-1} d_1(x,y) - b_1 \leq d_2(\phi(x), \phi(y)) \leq a_1 d_1(x,y) + b_1$

$\exists\ a_2, b_2$ s.t. $a_2^{-1} d_2(\tilde{x}, \tilde{y}) - b_2 \leq d_3(\psi(\tilde{x}), \psi(\tilde{y})) \leq a_2 d_2(\tilde{x}, \tilde{y}) + b_2$

(a) $$a_2^{-1} d_2(\phi(x), \phi(y)) - b_2 \le d_3(\psi\circ\phi(x), \psi\circ\phi(x))$$

$$\le$$

$$a_2^{-1}(a_1^{-1} d(x,y) - b_1) - b_2$$

$$\le a_2 d_2(\phi(x), \phi(y)) + b_2$$

$$||$$

$$\le a_2 a_1 d_1(x,y) + a_2 b_1 + b_2$$

$$(a_1 a_2)^{-1} d(x,y) - a_2^{-1} b_1 - b_2$$

(b), Let $z \in Z$, $\exists\ y \in Y$ s.t.

$$d_3(z, \psi(y)) \le c_2$$

$\exists\ x \in X$ s.t.

$$d_2(y, \phi(x)) \le c_1$$

$$d_3(z, \psi\circ\phi(x)) \leq d_3(z, \psi(y)) + d_3(\psi(y), \psi\circ\phi(x))$$

$$\leq c_2 + (a_2 d_2(y, \phi(x)) + b_2)$$

$$\leq c_2 + a_2 c_1 + b_2$$

(2). Symmetric: Given $\phi : (X_1, d_1) \to (X_2, d_2)$ rough isometry

Want to find

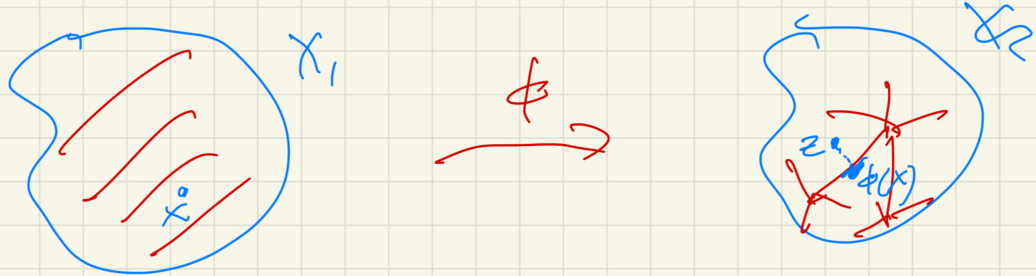
$\bar{\Phi} : (X_2, d_2) \to (X_1, d_1)$ rough isometry.

Pf: Pick any $z \in X_2$.

$\exists\ x \in X_1$ s.t.

$$d_2(z, \phi(x)) < C$$

Define $\bar{\Phi}(z) = x$.



Rmk: $\bar{\Phi}$ is not unique, not continuous generally

Ex: $\bar{\Phi}$ is rough isometry.

Def Γ_1, Γ_2 locally finite graphs are roughly isometric if

(V_1, d_{com}), (V_2, d_{com}) roughly isometric.

Rmk: In this case, all resistance $\equiv 1$.

Goal: If Γ_1, Γ_2 with bounded vertex degree are roughly isometric, then Γ_1 is recurrent $\Leftrightarrow$ Γ_2 is recurrent.

$\Gamma_1 \in O_{HD} \Leftrightarrow \Gamma_2 \in O_{HD}$.