On homotopy braids

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1. Introduction

Classical braid group can be defined as the fundamental group of configuration space or as the mapping class group of a disc with n punctures. Being a natural object, braids admit generalizations in various directions. Also there are special types of braids defined among all braids by specific properties.

Two geometric braids with the same endpoints are called *homotopic* if one can be deformed to the other by homotopies of the braid strings which fix the endpoints, so that different strings do not intersect. If two geometric braids are isotopic, they are evidently homotopic.

E. Artin posed the question of whether the notions of isotopy and homotopy of braids are different or the same. Namely he wrote:

''Assume that two braids can be deformed into each other by a deformation of the most general nature including self intersection of each string but avoiding intersection of two different strings. Are they isotopic?"

Deborah Goldsmith gave an example of a braid which is not trivial in the isotopic sense, but is homotopic to the trivial braid. At first she expressed this braid and homotopy process by the pictures. We give these pictures in Figure 1.

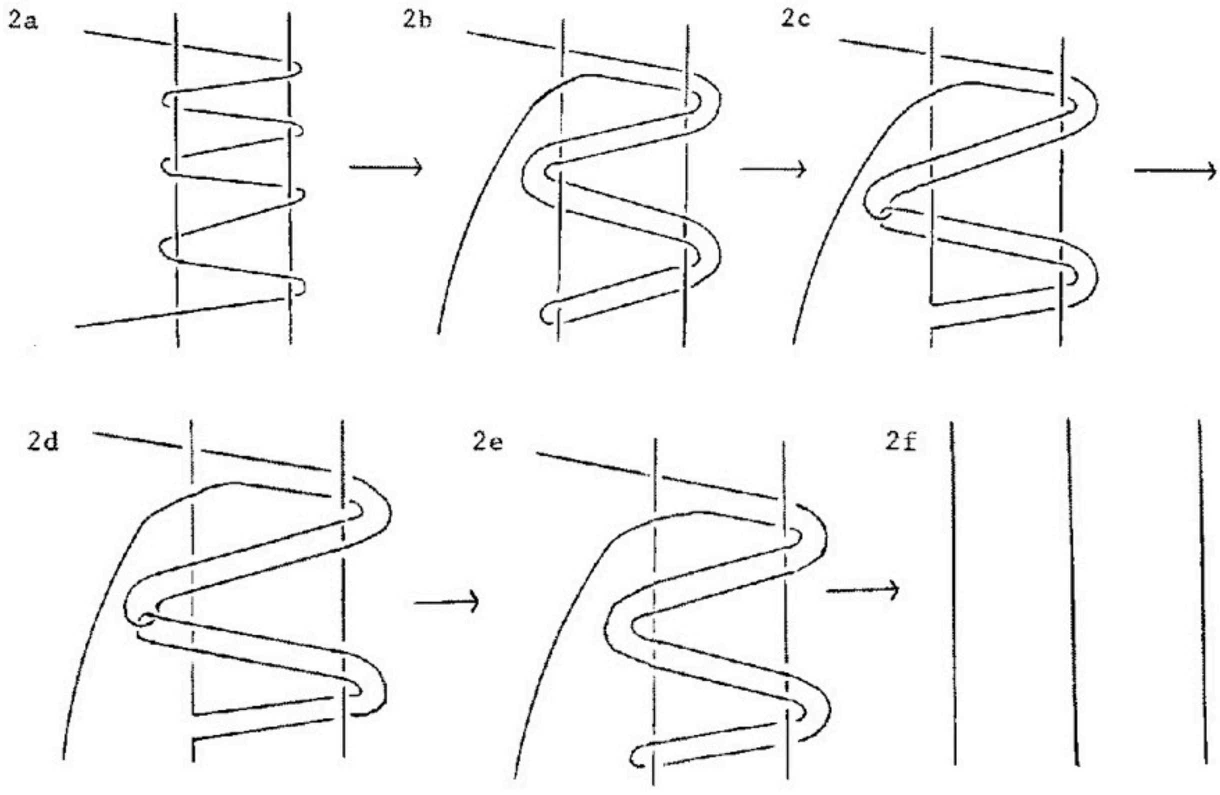

Figure: 1

This braid is expressed in the canonical generators of the classical braid group in the following form:

$$\sigma_1\sigma_2^2\sigma_1^2\sigma_2^{-2}\sigma_1^{-2}\sigma_2^2\sigma_1^{-2}\sigma_2^{-2}\sigma_1.$$

2. Artin presentation for braid group

Artin presentation of the braid group Br_n has generators σ_i, $i = 1, ..., n-1$ and relations:

$$\begin{cases} \sigma_i \sigma_j & = \sigma_j \sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i & = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{cases}$$

3. Presentaion of the pure braid group

Define the elements $a_{i,j}$, $1 \leq i < j \leq n$, of Br_n by:

$$a_{i,j} = \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \ldots \sigma_{j-1}^{-1}.$$

Geometrically generator of this type is depicted as follows

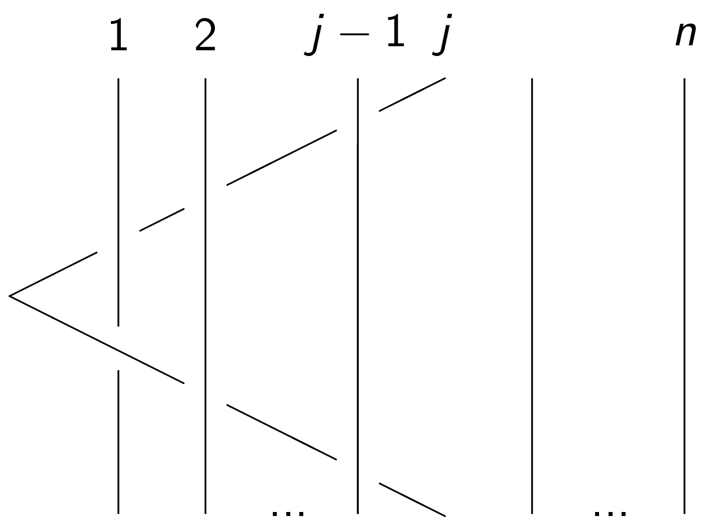

Figure: Generator $a_{1,j}$

They satisfy the Burau relations:

$$
\begin{aligned}
a_{i,j}a_{k,l} = a_{k,l}a_{i,j} \text{ for } i<j<k<l \text{ and } i<k<l<j,\\
a_{i,j}a_{i,k}a_{j,k} = a_{i,k}a_{j,k}a_{i,j} \text{ for } i<j<k,\\
a_{i,k}a_{j,k}a_{i,j} = a_{j,k}a_{i,j}a_{i,k} \text{ for } i<j<k,\\
a_{i,k}a_{j,k}a_{j,l}a_{j,k}^{-1} = a_{j,k}a_{j,l}a_{j,k}^{-1}a_{i,k} \text{ for } i<j<k<l.
\end{aligned}
\tag{1}
$$

W. Burau proved that this gives a presentation of the pure braid group P_n.

4. Reduced free group

For elements a, b of arbitrary group G we will use the following notations

$$a^b = b^{-1}ab, \quad [a, b] = a^{-1}b^{-1}ab.$$

Let $F_n = F(x_1, \ldots, x_n)$ be the free group on generators $x_1, \ldots, x_n$. We denote by K_n the quotient group of F_n by the relations

$$[x_i, x_i^g] = 1, \ i = 1, \ldots, n,$$

where g is an arbitrary element of F_n. The group K_n is called the *reduced free group*.

It is the quotient group of the free group obtained by adding relations which express that each x_i commutes with all of its conjugates.

This group can be characterized also the following way. Let X_i be the normal subgroup of F_n generated by x_i and let $[X_i]$ be the commutator subgroup of X_i. Then $N_n = [X_1] \dots [X_n]$ is also the normal subgroup of F_n and K_n is the quotient group F_n/N_n.

This group was introduced by J. Milnor and studied by Habegger & Lin, F. Cohen and F. Cohen & Jie Wu.

5. Homotopy braid group

Recall that the homotopy braid group $\widehat{B}_n$ is the quotient of the braid group B_n by the relations

$$[a_{ik}, a_{ik}^{g}] = 1, \text{ where } g \in \langle a_{1k}, a_{2k}, \ldots, a_{k-1,k} \rangle, 1 \leq i < k \leq n.$$

Let us denote by ϕ the canonical epimorphism from the standard braid group to the homotopy braid group

$$\phi : B_n \to \widehat{B}_n.$$

The quotient of the pure braid group P_n by the same relations gives us the pure homotopy braid group $\widehat{P}_n$ and from the standard short exact sequence for B_n we have the following short exact sequence

$$1 \longrightarrow \widehat{P}_n \longrightarrow \widehat{B}_n \longrightarrow S_n \longrightarrow 1,$$

where S_n is the symmetric group.

The group $\widehat{P}_n$ has the decomposition $\widehat{P}_n = \widehat{U}_n \rtimes \widehat{P}_{n-1}$, where $\widehat{U}_n$ is the quotient of the free group $U_n = \langle a_{1n}, a_{2n}, \ldots, a_{n-1,n} \rangle$ of rank $n-1$ by the relations

$$[a_{in}, a_{in}^{g}] = 1, \text{ where } g \in U_n, 1 \leq i < k \leq n.$$

Note, that $\widehat{U}_n$ is isomorphic to K_{n-1}. In particular, $\widehat{U}_2$ is isomorphic to the infinite cyclic group and $\widehat{U}_3$ is the quotient of $U_3 = \langle a_{13}, a_{23} \rangle$ by the relations

$$a_{13} \cdot a_{23}^{-1} a_{13} a_{23} = a_{23}^{-1} a_{13} a_{23} \cdot a_{13},$$

$$a_{23} \cdot a_{13}^{-1} a_{23} a_{13} = a_{13}^{-1} a_{23} a_{13} \cdot a_{23}.$$

The canonical Artin monomorphism

$$\nu_n : B_n \hookrightarrow \mathrm{Aut}\, F_n$$

induces a monomorphism

$$\hat{\nu}_n : \widehat{B}_n \to \mathrm{Aut}\, K_n.$$

(Cohen and Wu).

Theorem (Habegger and Lin)

K_n is a finitely generated nilpotent group of class n.

6. Linearity

6.1. Existence

Recall that a group G is called *linear* if it has a faithful representation into the general linear group $GL_m(k)$ for some m and a field k.

Theorem

The homotopy braid group $\widehat{B}_n$ is linear for all $n \geq 2$. Moreover, for every $n \geq 2$ there is a natural m such that there exists a faithful representation

$$\widehat{B}_n \longrightarrow GL_m(\mathbb{Z}).$$

Proof. The reduced free group K_n, $n \geq 2$ is nilpotent. Finitely generated nilpotent groups are polycyclic and hence they are represented by integer matrices as was proved by L.Auslender and R.G.Swan. Also the holomorph of every polycyclic group has a faithful representation into $GL_m(\mathbb{Z})$ for some m. Hence, the holomorph Hol(K_n) has a faithful representation into $GL_m(\mathbb{Z})$ for some m. And Hol(K_n) contains Aut(K_n) as a subgroup and as $\widehat{B}_n$ is embedded into Aut(K_n) there exists a monomorphism $\widehat{B}_n \longrightarrow GL_m(\mathbb{Z})$. □

6.2. Homotopy braids and the Burau representation

It is interesting to find a faithful linear representation of $\widehat{B}_n$ explicitly. For example, is it possible to do with the help of the Burau representation?

Let

$$\rho_B : B_n \longrightarrow GL(W_n)$$

be the Burau representation of B_n, where W_n is a free $\mathbb{Z}[t^{\pm 1}]$-module of rank n with the basis $w_1, w_2, \ldots, w_n$.

Let $n = 3$. In this case the automorphisms $\rho_B(\sigma_i)$, $i = 1, 2$, of module W_3 act by the rule

$$\sigma_1 : \begin{cases} w_1 \longmapsto (1-t)w_1 + tw_2, \\ w_2 \longmapsto w_1, \\ w_3 \longmapsto w_3, \end{cases} \qquad \sigma_2 : \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto (1-t)w_2 + tw_3, \\ w_3 \longmapsto w_2, \end{cases}$$

where we write for simplicity σ_i instead of $\rho_B(\sigma_i)$.

Let us find the action of the generators of P_3 on the module W_3. Recall, that $P_3 = U_3 \rtimes U_2$, where U_2 is the infinite cyclic group with the generator $a_{12} = \sigma_1^2$, U_3 is the free group of rank 2 with the free generators

$$a_{13} = \sigma_2 \sigma_1^2 \sigma_2^{-1}, \quad a_{23} = \sigma_2^2.$$

These elements define the following automorphisms of W_3

$$a_{12} : \begin{cases} w_1 \longmapsto (1 - t + t^2)w_1 + t(1 - t)w_2, \\ w_2 \longmapsto (1 - t)w_1 + tw_2, \\ w_3 \longmapsto w_3, \end{cases} \tag{2}$$

$$
a_{13} : \begin{cases} w_1 \longmapsto (1-t+t^2)w_1 + t(1-t)w_3, \\ w_2 \longmapsto (1-t)^2 w_1 + w_2 - (1-t)^2 w_3, \\ w_3 \longmapsto (1-t)w_1 + tw_3, \end{cases} \tag{3}
$$

$$
a_{23}: \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto (1 - t + t^2) w_2 + t(1 - t) w_3, \\ w_3 \longmapsto (1 - t) w_2 + t w_3, \end{cases} \tag{4}
$$

$$a_{23}^{-1} : \begin{cases} w_1 \longmapsto w_1, \\ w_2 \longmapsto t^{-1} w_2 + (1 - t^{-1}) w_3, \\ w_3 \longmapsto t^{-1}(1 - t^{-1}) w_2 + (1 - t^{-1} + t^{-2}) w_3. \end{cases} \quad (5)$$

Let us denote by $\widehat{\rho}_B$ the representation (if it exists)

$$\widehat{\rho}_B : \widehat{B}_n \longrightarrow GL(W_n)$$

such that

$$\rho_B = \widehat{\rho}_B \circ \phi : B_n \to GL(W_n). \tag{6}$$

Proposition

For $n = 3$ the representation $\widehat{\rho}_B$ such that the condition (6) holds exists only if we consider the specialization of the Burau representation with $t = 1$. In this case $\widehat{\rho}_B$ is trivial on $\widehat{P}_3$. Hence, the image $\widehat{\rho}_B(\widehat{B}_3)$ is isomorphic to the symmetric group S_3.

Proof. To obtain a representation $\widehat{\rho}_B(\widehat{B}_3)$ we must have the following relations among the automorphisms $a_{i,j}$ (2)-(4) of W_3:

$$[a_{13}, a_{13}^{a_{23}}] = 1, \quad [a_{23}, a_{23}^{a_{13}}] = 1,$$

which are equivalent to the following relations

$$a_{13} a_{13}^{a_{23}} = a_{13}^{a_{23}} a_{13}, \quad a_{23} a_{23}^{a_{13}} = a_{23}^{a_{13}} a_{23}.$$

From the definitions of automorphisms (2)-(5) we obtain

$$a_{23}^{-1} a_{13} a_{23} : \begin{cases} w_1 \longmapsto (1 - t + t^2) w_1 + t(1 - t)^2 w_2 + t^2(1 - t) w_3, \\ w_2 \longmapsto w_2, \\ w_3 \longmapsto t^{-1}(1 - t) w_1 - t^{-1}(1 - t)^2 w_2 + t w_3. \end{cases}$$

$$
a_{13}a_{13}^{a_{23}} : \begin{cases} w_1 \longmapsto & (2-4t+4t^2-2t^3+t^4)w_1 + \\ & (1-t)^2(-1-t^2+t^3)w_2 + \\ & t^2(1-t)(2-t+t^2)w_3, \\ w_2 \longmapsto & (1-t)^2(-t^{-1}+2-t+t^2)w_1 + \\ & [(1-t)^4(t+t^{-1})+1]w_2 + \\ & t(1-t)^2(-1+t-t^2)w_3, \\ w_3 \longmapsto & (1-t)(2-t+t^2)w_1 + \\ & (1-t)^2[-1+t-t^2]w_2 + \\ & +t^2(2-2t+t^2)w_3. \end{cases}
$$

$$a_{13}^{a_{23}} a_{13} : \begin{cases} w_1 \longmapsto & (1-t+2t^3-2t^4+t^5)w_1 + t(1-t)^2 w_2 + \\ & +t(1-t)(1-2t+5t^2-3t^3+t^4)w_3, \\ w_2 \longmapsto & (1-t)^2 w_1 + w_2 - (1-t)^2 w_3, \\ w_3 \longmapsto & (1-t)(2-t+t^2)w_1 - t^{-1}(1-t)^2 w_2 + \\ & +[(1-t)^2(1+t-2t^2+t^3)+t^2]w_3. \end{cases}$$

In order to satisfy relation $a_{13}a_{13}^{a_{23}} = a_{13}^{a_{23}}a_{13}$ the following system of equations should have a solution

$$\begin{cases} 1-3t+4t^2-4t^3+3t^4-t^5=0, \\ (1-t)^2(-1-t-t^2+t^3)=0, \\ t(1-t)^5=0, \\ (1-t)^2(-t^{-1}+1-t+t^2)=0, \\ t^{-1}(1-t)^4(1+t^2)=0, \\ (1-t)^2(1-t+t^2-t^3)=0, \\ (1-t)^2(-1+t-t^2+t^{-1})=0, \\ 1-t-4t^2+8t^3-5t^4+t^5=0. \end{cases}$$

This system has a solution only if $t = 1$. In this case, automorphisms a_{12}, a_{13}, a_{23} are equal to the identity automorphism. $\square$

7. Torsion in $\widehat{B}_n$

V.Ya. Lin formulated the following question in the Kourovka Notebook

Question

Is there a non-trivial epimorphism of B_n onto a non-abelian group without torsion?

An answer to this question was given by P. Linnell and T. Schick in 2007.

We conjecture that the group $\widehat{B}_n$, $n \geq 3$, does not have torsion and since there exists the epimorphism $B_n \longrightarrow \widehat{B}_n$, the group $\widehat{B}_n$ will be another example that answers Lin's question.

We prove that $\widehat{B}_3$ does not have torsion.

Let $\widehat{P}_3$, $\widehat{U}_2$, $\widehat{U}_3$ be the images of P_3, U_2, U_3 by the canonical epimorphism $B_3 \longrightarrow \widehat{B}_3$. Denote by b_{ij}, $1 \leq i < j \leq 3$ the images of a_{ij}, $1 \leq i < j \leq 3$, by this epimorphism. Then $\widehat{U}_2 = \langle b_{12} \rangle$ is the infinite cyclic group and

$$\widehat{U}_3 = \langle b_{13}, b_{23} \mid\mid [b_{13}, b_{13}^{b_{23}}] = [b_{23}, b_{23}^{b_{13}}] = 1 \rangle =$$

$$= \langle b_{13}, b_{23} \mid\mid [b_{13}, b_{13}[b_{13}, b_{23}]] = [b_{23}, b_{23}[b_{23}, b_{13}]] = 1 \rangle.$$

Using commutator identities or direct calculations we see that the last two relations are equivalent to the following relation

$$[[b_{23}, b_{13}], b_{23}] = [[b_{23}, b_{13}], b_{13}] = 1.$$

Hence, $\widehat{U}_3$ is a free 2-step nilpotent group of rank 2 and so, every element $g \in \widehat{U}_3$ has a unique presentation of the form

$$g = b_{13}^{\alpha} b_{23}^{\beta} [b_{23}, b_{13}]^{\gamma}$$

for some integers α, β, γ.

The same way as in the case of classical braid group, $\widehat{U}_3$ is a normal subgroup of $\widehat{P}_3$ and the action of $\widehat{U}_2$ is defined in the following lemma.

Lemma

The action of $\widehat{U}_2$ on $\widehat{U}_3$ is given by the formulas

$$b_{13}^{b_{12}^k} = b_{13}[b_{23}, b_{13}]^k, \quad b_{23}^{b_{12}^k} = b_{23}[b_{23}, b_{13}]^{-k}, \quad [b_{23}, b_{13}]^{b_{12}^k} = [b_{23}, b_{13}], \quad k$$

The action of the generators σ_1 and σ_2 of $\widehat{B}_3$ on $\widehat{P}_3$ is given in the next lemma.

Lemma

The following conjugation formulas hold in $\widehat{B}_3$

$$b_{12}^{\sigma_1^{\pm 1}} = b_{12}, \quad b_{13}^{\sigma_1} = b_{23}[b_{23}, b_{13}]^{-1}, \quad b_{23}^{\sigma_1} = b_{13}, b_{13}^{\sigma_1^{-1}} = b_{23},$$

$$b_{23}^{\sigma_1^{-1}} = b_{13}[b_{23}, b_{13}]^{-1}, [b_{23}, b_{13}]^{\sigma_1^{-1}} = [b_{23}, b_{13}]^{-1},$$

$$b_{12}^{\sigma_2} = b_{13}[b_{23}, b_{13}]^{-1}, \quad b_{13}^{\sigma_2} = b_{12}, \quad b_{23}^{\sigma_2^{\pm 1}} = b_{23}, \quad b_{12}^{\sigma_2^{-1}} = b_{13},$$

$$b_{13}^{\sigma_2^{-1}} = b_{12}[b_{23}, b_{13}]^{-1}, [b_{23}, b_{13}]^{\sigma_2^{-1}} = [b_{23}, b_{13}]^{-1}. \quad \square$$

Let us denote by $\Lambda_3 = \{e, \sigma_1, \sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_2\sigma_1\}$ the set of representatives of $\widehat{P}_3$ in $\widehat{B}_3$. Then every element in $\widehat{B}_3$ can be written in the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \lambda, \text{ where } \alpha, \beta, \gamma, \delta \in \mathbb{Z}, \quad z = [b_{23}, b_{13}], \quad \lambda \in \Lambda_3.$$

Theorem

The group $\widehat{B}_3$ *is torsion-free.*

Proof. The group $\widehat{P}_3$ does not have torsion. Hence, if $\widehat{B}_3$ has elements of finite order, then they have the form

$$b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \lambda, \quad \lambda \in \Lambda_3 \setminus \{e\}.$$

Every element which is conjugate with an element of finite order has a finite order. Taking into account the following formulas

$$\sigma_1^{-1}\cdot\sigma_2\cdot\sigma_1 = b_{12}^{-1}\sigma_1\sigma_2\sigma_1,\ \ \sigma_2\sigma_1\cdot\sigma_2\cdot\sigma_1^{-1}\sigma_2^{-1} = \sigma_1,\ \ \sigma_1^{-1}\cdot\sigma_1\sigma_2\cdot\sigma_1 = \sigma_2\sigma_1,$$

it is sufficient to consider only two cases: $\lambda = \sigma_2$ and $\lambda = \sigma_1\sigma_2$.

Let $\lambda = \sigma_2$, take $g = b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \sigma_2$. Then we have

$$g^2 = b_{12}^{\alpha+\beta} b_{13}^{\alpha+\beta} b_{23}^{2\gamma+1} z^{\alpha\gamma+\beta(\beta-\gamma+\alpha-1)}.$$

If $g^2 = 1$, then $\alpha + \beta = 0$ and we have

$$g^2 = b_{23}^{2\gamma+1} z^{2\alpha\gamma+\alpha}.$$

Since $2\gamma + 1$ cannot be zero for integer γ, the elements of this form cannot be of finite order.

Let $\lambda = \sigma_1\sigma_2$. Then we have

$$(\sigma_1\sigma_2)^2 = b_{12}\sigma_2\sigma_1, \quad (\sigma_1\sigma_2)^3 = b_{12}b_{13}b_{23}.$$

We calculate

$$g^3 = (b_{12}^{\alpha} b_{13}^{\beta} b_{23}^{\gamma} z^{\delta} \sigma_1 \sigma_2)^3 =$$
$$b_{12}^{\alpha+\beta+\gamma+1} b_{13}^{\alpha+\beta+\gamma+1} b_{23}^{\alpha+\beta+\gamma+1} z^{\alpha(\alpha+2\gamma-\beta)+\beta^2+\gamma^2-\beta\gamma+3\delta+3\beta}.$$

If $g^3 = 1$, then the following system of linear equations has a solution over $\mathbb{Z}$

$$\begin{cases} \alpha + \beta + \gamma + 1 = 0, \\ \alpha(\alpha + 2\gamma - \beta) + \beta^2 + \gamma^2 - \beta\gamma + 3\delta + 3\beta = 0. \end{cases}$$

From the first equation one gets: $\alpha = -1 - \beta - \gamma$. Inserting this equality into the second equation, we have

$$3(\beta^2 + 2\beta + \delta) + 1 = 0.$$

However, this equation does not have integer solutions. $\square$