Suspension splittings of manifolds and their applications

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Introduction

Assume all spaces are pointed CW-complexes and all maps are pointed and cellular.

We say that a space X admits a suspension splitting if $\Sigma X \simeq \bigvee Y_\alpha$.
Examples:

- $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma X \wedge Y$
- $\Sigma(\prod_{i=1}^m X_i) \simeq \bigvee_{\{i_1,\ldots,i_k\}\subseteq[m]} \Sigma X_{i_1} \wedge \cdots \wedge X_{i_k}$
- $\Sigma(\underline{X}, \underline{A})^K \simeq \bigvee_{I\subseteq[m]} \Sigma(\widehat{\underline{X}, \underline{A}})^{K_I}$
- $\Sigma\Omega S^{m+1} \simeq \bigvee_{i=1}^\infty S^{mi+1}$
- $\Sigma M \simeq \bigvee_{i=1}^{2g} S^2 \vee S^3$ if M is an orientable, closed, connected 2-dim manifold

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- $\Sigma(\prod_{i=1}^m X_i) \simeq \bigvee_{\{i_1,\ldots,i_k\}\subseteq[m]} \Sigma X_{i_1} \wedge \cdots \wedge X_{i_k}$
- $\Sigma(\underline{X},\underline{A})^K \simeq \bigvee_{I\subseteq[m]} \Sigma(\widehat{\underline{X},\underline{A}})^{K_I}$
- $\Sigma\Omega S^{m+1} \simeq \bigvee_{i=1}^\infty S^{mi+1}$
- $\Sigma M \simeq \bigvee_{i=1}^{2g} S^2 \vee S^3$ if M is an orientable, closed, connected 2-dim manifold

Classification of 2-dim manifolds: $M \cong \#^g S^1 \times S^1 \cong \boxed{\bigvee_{i=1}^{2g} S^1} \cup \boxed{e^2}$

i	0	1	2	≥ 3
$H_i(M)$	$\mathbb{Z}$	$\boxed{\bigoplus_{i=1}^{2g} \mathbb{Z}}$	$\boxed{\mathbb{Z}}$	0

Application I: Generalized cohomology theory

A reduced generalized cohomology theory $\{E^n(-) : CW_* \to Ab\}$ is a sequence of contravariant functors satisfying

- $E^n(X) \cong E^{n+1}(\Sigma X)$;
- If $f \simeq g : X \to Y$ then $f^* = g^* : E^n(Y) \to E^n(X)$;
- For inclusion $A \hookrightarrow X$ there is an exact sequence

$$E^n(X/A) \to E^n(X) \to E^n(A);$$

- For any wedge sum $X = \bigvee X_\alpha$ the inclusions $\imath_\alpha : X_\alpha \hookrightarrow X$ induce an isomorphism

$$\prod \imath_\alpha^* : E^n(X) \to \prod E^n(X_\alpha).$$

Examples: singular cohomology, topological K-theory, cobordism theories, etc

Proposition

Let $E^*(-)$ be a reduced generalized cohomology theory, and let M be a 2-dim orientable closed manifold. Then there is a group isomorphism

$$E^*(M) \cong \boxed{\bigoplus_{i=1}^{2g} E^*(S^1)} \oplus \boxed{E^*(S^2)}.$$

Proof: Let $E^n(-) \cong [-, K_n]$ for some Ω-spectrum $\{K_n\}$. Then

$$\begin{aligned}
E^n(M) &\cong [M, K_n] \\
&\cong [\Sigma M, K_{n+1}] \\
&\cong [\bigvee_{i=1}^{2g} S^2 \vee S^3, K_{n+1}] \\
&\cong \bigoplus_{i=1}^{2g} [S^2, K_{n+1}] \oplus [S^3, K_{n+1}] \\
&\cong \bigoplus_{i=1}^{2g} [S^1, K_n] \oplus [S^2, K_n] \\
&\cong \bigoplus_{i=1}^{2g} E^n(S^1) \oplus E^n(S^2)
\end{aligned}$$

Application II: Gauge groups

- Let G be a topological group.

- Let $\pi : P \to M$ be a principal G-bundle over M.

$$\begin{array}{ccc} \{\text{isom classes of prin } G\text{-bundles}\} & \xleftrightarrow{1:1} & [M, BG] \\ \pi : P \to M & & \alpha \end{array}$$

- The gauge group $\mathcal{G}_\alpha(M; G)$ of P is the topological group of G-equivariant automorphisms of P that fix M

$$\mathcal{G}_\alpha(M; G) = \left\{ \varphi \in \mathrm{Aut}(P) \;\middle|\; \begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ P & \xrightarrow{\varphi} & P \end{array} \right\}$$

- The topology of gauge groups relates to the Donaldson's Theory, the homotopy theory of moduli spaces of stable vector bundles, etc

If G is a simple, connected, compact Lie group, then $[M, BG] \cong \mathbb{Z}$ and P is classified by its first Chern class $\alpha \in \mathbb{Z}$.

Proposition

Let G be a simple, connected, compact Lie group and let M be an orientable closed 2-dim manifold. Let P be a principal G-bundle with first Chern class α. Then there is a homotopy equivalence

$$\mathcal{G}_\alpha(M; G) \quad \simeq \quad \boxed{\mathcal{G}_\alpha(S^2; G)} \times \boxed{\bigoplus_{i=1}^{2g} \Omega G}.$$

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Theorem (Sutherland,92)

Let M be a closed Riemann surface and let $G = U(n)$. For any integers α, β

1. *if $\mathcal{G}_\alpha(M, U(n)) \simeq \mathcal{G}_\beta(M, U(n))$ then $gcd(\alpha, n) = gcd(\beta, n)$;*
2. *if $gcd(\alpha, n) = gcd(\beta, n)$ then $\mathcal{G}_\alpha(M, U(n)) \simeq \mathcal{G}_\beta(M, U(n))$ after completion at any prime.*

4-dimensional smooth manifolds

M: an orientable, closed, connected, smooth 4-manifold

i	0	1	2	3	4
$H_i(M)$	$\mathbb{Z}$	$\boxed{\mathbb{Z}^m} \oplus \boxed{\bigoplus_{j=1}^{l} \mathbb{Z}/t_j}$	$\boxed{\mathbb{Z}^n} \oplus \boxed{\bigoplus_{j=1}^{l} \mathbb{Z}/t_j}$	$\boxed{\mathbb{Z}^m}$	$\boxed{\mathbb{Z}}$

where $\bigoplus_{j=1}^{l} \mathbb{Z}/t_j\mathbb{Z}$ is the torsion part of $H_1(M)$ and $H_2(M)$.

Theorem (S.-Theriault)

Let M be an orientable, closed, connected, smooth 4-manifold. Suppose $H_1(M)$ has no 2-torsion elements. If M is spin ($Sq^2 : H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ is trivial), then

$$\Sigma M \simeq \boxed{\bigvee_{i=1}^{m} S^2} \vee \boxed{\bigvee_{j=1}^{l} P^3(t_j)} \vee \boxed{\bigvee_{k=1}^{n} S^3} \vee \boxed{\bigvee_{j=1}^{l} P^4(t_j)} \vee \boxed{\bigvee_{i=1}^{m} S^4} \vee \boxed{S^5}.$$

If M is non-spin ($Sq^2 : H^2(M;\mathbb{Z}/2) \to H^4(M;\mathbb{Z}/2)$ is non-trivial), then

$$\Sigma M \simeq \boxed{\bigvee_{i=1}^{m} S^2} \vee \boxed{\bigvee_{j=1}^{l} P^3(t_j)} \vee \boxed{\bigvee_{k=1}^{n-1} S^3} \vee \boxed{\bigvee_{j=1}^{l} P^4(t_j)} \vee \boxed{\bigvee_{i=1}^{m} S^4} \vee \boxed{\Sigma\mathbb{C}P^2}.$$

where $P^{i+1}(t) = S^i \cup_t e^{i+1}$ is the mod-t Moore space

Application I: Generalized cohomology theory

Theorem (S.-Theriault)

Let $E^*(-)$ be a reduced generalized cohomology theory. If M is spin, then there is a group isomorphism

$$\begin{aligned} E^*(M) \quad \cong \quad & \boxed{E^*(S^4)} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^3)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^3(t_j))} \oplus \\ & \boxed{\bigoplus_{k=1}^{n} E^*(S^2)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^2(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^1)}. \end{aligned}$$

If M is non-spin, then

$$\begin{aligned} E^*(M) \quad \cong \quad & \boxed{E^*(\mathbb{CP}^2)} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^3)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^3(t_j))} \oplus \\ & \boxed{\bigoplus_{k=1}^{n-1} E^*(S^2)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^2(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^1)}. \end{aligned}$$

Application II: Gauge groups

Suppose G is a simple, simply connected, compact Lie group. Then

- $[M, BG] \cong H^4(M) \cong \mathbb{Z}$
- P is classified by its second Chern class α.

Theorem (S.-Theriault)

Assume M satisfies the conditions of the Theorem. If M is spin there is a homotopy equivalence

$$\begin{aligned}\mathcal{G}_\alpha(M;G) \quad \simeq \quad & \boxed{\mathcal{G}_\alpha(S^4;G)} \times \boxed{\prod_{i=1}^{m} \Omega^3 G} \times \boxed{\prod_{j=1}^{l} \Omega^3 G\{t_j\}} \times \\ & \boxed{\prod_{k=1}^{n} \Omega^2 G} \times \boxed{\prod_{j=1}^{l} \Omega^2 G\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega G}.\end{aligned}$$

If M is non-spin and $\pi_1(M)$ is a graph product of $\mathbb{Z}$ and $\mathbb{Z}/t_j\mathbb{Z}$, then

$$\begin{aligned}\mathcal{G}_\alpha(M;G) \quad \simeq \quad & \boxed{\mathcal{G}_\alpha(\mathbb{CP}^2;G)} \times \boxed{\prod_{i=1}^{m} \Omega^3 G} \times \boxed{\prod_{j=1}^{l} \Omega^3 G\{t_j\}} \times \\ & \boxed{\prod_{k=1}^{n-1} \Omega^2 G} \times \boxed{\prod_{j=1}^{l} \Omega^2 G\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega G}.\end{aligned}$$

where $\Omega^n G = \mathrm{Map}^*(S^n, G)$ and $\Omega^n G\{k\} = \mathrm{Map}^*(P^n(k), G)$.

Example

Let $M = \mathbb{CP}^2 \# (S^1 \times S^3)$. Its homology groups are

i	0	1	2	3	4
$H_i(M)$	$\mathbb{Z}$	$\boxed{\mathbb{Z}}$	$\boxed{\mathbb{Z}}$	$\boxed{\mathbb{Z}}$	$\boxed{\mathbb{Z}}$

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As M is non-spin we have

$$\begin{aligned}
\Sigma M &\simeq \boxed{\Sigma\mathbb{CP}^2} \vee \boxed{S^4} \vee \boxed{S^2} \\
E^*(M) &\cong \boxed{E^*(\mathbb{CP}^2)} \oplus \boxed{E^*(S^3)} \vee \boxed{E^*(S^1)} \\
\mathcal{G}_\alpha(M; G) &\simeq \boxed{\mathcal{G}_\alpha(\mathbb{CP}^2; G)} \times \boxed{\Omega^3 G} \times \boxed{\Omega G}
\end{aligned}$$

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E^*(M) &\cong \boxed{E^*(\mathbb{CP}^2)} \oplus \boxed{E^*(S^3)} \vee \boxed{E^*(S^1)} \\
\mathcal{G}_\alpha(M; G) &\simeq \boxed{\mathcal{G}_\alpha(\mathbb{CP}^2; G)} \times \boxed{\Omega^3 G} \times \boxed{\Omega G}
\end{aligned}$$

Topological $\tilde{K}$-theory: $\tilde{K}^0(M) \cong \boxed{\mathbb{Z}^2}$ and $\tilde{K}^1(M) \cong \boxed{\mathbb{Z}} \oplus \boxed{\mathbb{Z}}$

6-dimensional smooth manifolds

M: orientable, closed, simply connected, smooth 6-manifold

i	0	1	2	3	4	5	6
$H_i(M)$	$\mathbb{Z}$	0	$\boxed{\mathbb{Z}^m} \oplus \boxed{T}$	$\boxed{\mathbb{Z}^{2n}} \oplus \boxed{T}$	$\boxed{\mathbb{Z}^m}$	0	$\boxed{\mathbb{Z}}$

where $T = \bigoplus_{j=1}^{\ell} \mathbb{Z}/t_j\mathbb{Z}$ is the torsion part of $H_2(M)$

Theorem (Cutler-S.)

Let M be an orientable, closed, simply-connected, smooth 6-manifold. Take localization away from 2. If $\mathcal{P}^1 : H^2(M;\mathbb{Z}/3) \to H^6(M;\mathbb{Z}/3)$ is trivial, then

$$\Sigma M \simeq_{(2)} \boxed{\bigvee_{i=1}^{m} S^3} \vee \boxed{\bigvee_{j=1}^{\ell} P^4(t_j)} \vee \boxed{\bigvee_{k=1}^{2n} S^4} \vee \boxed{\bigvee_{j=1}^{\ell} P^5(t_j)} \vee \boxed{\bigvee_{i=1}^{m} S^5} \vee \boxed{S^7}.$$

If $\mathcal{P}^1 : H^2(M;\mathbb{Z}/3) \to H^6(M;\mathbb{Z}/3)$ is non-trivial, then

$$\Sigma M \simeq_{(2)} \boxed{\bigvee_{i=1}^{m-1} S^3} \vee \boxed{\bigvee_{j=1}^{\ell} P^4(t_j)} \vee \boxed{\bigvee_{k=1}^{2n} S^4} \vee \boxed{\bigvee_{j=1}^{\ell} P^5(t_j)} \vee \boxed{\bigvee_{i=1}^{m-1} S^5} \vee \boxed{\Sigma\mathbb{CP}^3}$$

$$\Sigma M \simeq_{(2)} \boxed{\bigvee_{i=1}^{m} S^3} \vee \boxed{\bigvee_{\substack{1\leq j\leq \ell \\ j\neq \bar{c}}} P^4(t_j)} \vee \boxed{\bigvee_{k=1}^{2n} S^4} \vee \boxed{\bigvee_{j=1}^{\ell} P^5(t_j)} \vee \boxed{\bigvee_{i=1}^{m} S^5} \vee \boxed{C_{\bar{c}}}$$

where $C_{\bar{c}} \simeq P^4(3^r) \cup e^7$

i	0	1	2	3	4	5	6
$H_i(M)$	$\mathbb{Z}$	0	$\boxed{\mathbb{Z}^m} \oplus \boxed{T}$	$\boxed{\mathbb{Z}^{2n}} \oplus \boxed{T}$	$\boxed{\mathbb{Z}^m}$	0	$\boxed{\mathbb{Z}}$

Theorem (Huang)

Let M be an orientable, closed, simply-connected, smooth 6-manifold. Suppose $H_*(M)$ has no 2- or 3-torsion elements. Then

$$\Sigma^2 M \quad \simeq \quad \boxed{\bigvee_{i=1}^{m-t-1} S^4 \vee \bigvee_{s=1}^{t-1} \Sigma^2\mathbb{C}P^2} \vee \boxed{\bigvee_{j=1}^{l} P^5(t_j)} \vee \boxed{\bigvee_{k=1}^{2n} S^5} \vee \boxed{\bigvee_{j=1}^{l} P^6(t_j)} \vee \boxed{\bigvee_{i=1}^{m-t-1} S^6} \vee \boxed{\Sigma W_t}$$

for some number $t \in \{0, \ldots, m\}$, where

$$W_t \simeq \begin{cases} (S^3 \vee S^5) \cup e^7 & t = 0 \\ (\Sigma\mathbb{C}P^2 \vee S^3 \vee S^5) \cup e^7 & 1 \leq t \leq m-1 \\ \Sigma\mathbb{C}P^2 \cup e^7 & t = m \end{cases}$$

Application I: Generalized cohomology theory

Notation: For finitely generated abelian groups A, B we write

$$A \cong_{(2)} B \qquad \Leftrightarrow \qquad A \otimes \mathbb{Z}[\frac{1}{2}] \cong B \otimes \mathbb{Z}[\frac{1}{2}] \text{ where } \mathbb{Z}[\frac{1}{2}] = \{\frac{a}{2^r} \in \mathbb{Q}\}$$

Theorem (Cutler-S.)

Let $E^*(-)$ be a reduced generalized cohomology theory. If $\mathcal{P}^1 : H^2(M;\mathbb{Z}/3) \to H^6(M;\mathbb{Z}/3)$ is trivial, then there is a group isomorphism

$$\begin{aligned} E^*(M) \quad \cong_{(2)} \quad & \boxed{E^*(S^6)} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^4)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^4(t_j))} \oplus \\ & \boxed{\bigoplus_{k=1}^{2n} E^*(S^3)} \oplus \boxed{\bigoplus_{j=1}^{\ell} E^*(P^3(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^2)}. \end{aligned}$$

If $\mathcal{P}^1 : H^2(M;\mathbb{Z}/3) \to H^6(M;\mathbb{Z}/3)$ is non-trivial, then

$$\begin{aligned} E^*(M) \quad \cong_{(2)} \quad & \boxed{E^*(\mathbb{CP}^3)} \oplus \boxed{\bigoplus_{i=1}^{m-1} E^*(S^4)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^4(t_j))} \oplus \\ & \boxed{\bigoplus_{k=1}^{2n} E^*(S^3)} \oplus \boxed{\bigoplus_{j=1}^{\ell} E^*(P^3(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m-1} E^*(S^2)} \\ E^*(M) \quad \cong_{(2)} \quad & \boxed{E^{*+1}(C_{\bar{c}})} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^4)} \oplus \boxed{\bigoplus_{j=1}^{l} E^*(P^4(t_j))} \oplus \\ & \boxed{\bigoplus_{k=1}^{2n} E^*(S^3)} \oplus \boxed{\bigoplus_{\substack{1\le j\le \ell \\ j\neq \bar{c}}}^{\ell} E^*(P^3(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m} E^*(S^2)} \end{aligned}$$

Theorem (Huang)

Let $E^*(-)$ be a reduced generalized cohomology theory. Suppose $H_*(M)$ has no 2- or 3-torsion elements. Then

$$
\begin{aligned}
E^*(M) \quad \cong \quad & \boxed{E^{*+1}(W_t)} \oplus \boxed{\bigoplus_{s=1}^{t-1} E^*(\mathbb{CP}^2)} \oplus \boxed{\bigoplus_{i=1}^{m-t-1} E^*(S^4)} \oplus \\
& \boxed{\bigoplus_{j=1}^{l} E^*(P^4(t_j))} \oplus \boxed{\bigoplus_{k=1}^{2n} E^*(S^3)} \oplus \\
& \boxed{\bigoplus_{j=1}^{\ell} E^*(P^3(t_j))} \oplus \boxed{\bigoplus_{i=1}^{m-t-1} E^*(S^2)}.
\end{aligned}
$$

for some number $t \in \{0, \ldots, m\}$.

Application II: Gauge groups

Denote the gauge group of the trivial bundle $P = M \times G$ by $\mathcal{G}_0(M; G)$.

Theorem (Cutler-S.)

If $\mathcal{P}^1 : H^2(M; \mathbb{Z}/3) \to H^6(M; \mathbb{Z}/3)$ is trivial then

$$\begin{aligned}\mathcal{G}_0(M;G) \quad \simeq_{(2)} \quad & \boxed{G \times \Omega^6 G} \times \boxed{\prod_{i=1}^{m} \Omega^4 G} \times \boxed{\prod_{j=1}^{\ell} \Omega^4 G\{t_j\}} \times \\ & \boxed{\prod_{k=1}^{2n} \Omega^3 G} \times \boxed{\prod_{j=1}^{\ell} \Omega^3 G\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega^2 G}.\end{aligned}$$

If $\mathcal{P}^1 : H^2(M; \mathbb{Z}/3) \to H^6(M; \mathbb{Z}/3)$ is non-trivial, then

$$\begin{aligned}\mathcal{G}_0(M;G) \quad \simeq_{(2)} \quad & \boxed{G \times \mathrm{Map}^*(\mathbb{CP}^3, G)} \times \boxed{\prod_{i=1}^{m-1} \Omega^4 G} \times \boxed{\prod_{j=1}^{\ell} \Omega^4 G\{t_j\}} \times \\ & \boxed{\prod_{k=1}^{2n} \Omega^3 G} \times \boxed{\prod_{j=1}^{\ell} \Omega^3 G\{t_j\}} \times \boxed{\prod_{i=1}^{m-1} \Omega^2 G} \\ \mathcal{G}_0(M;G) \quad \simeq_{(2)} \quad & \boxed{G \times \mathrm{Map}^*(C_{\bar{c}}, BG)} \times \boxed{\prod_{i=1}^{m} \Omega^4 G} \times \boxed{\prod_{j=1}^{\ell} \Omega^4 G\{t_j\}} \times \\ & \boxed{\prod_{k=1}^{2n} \Omega^3 G} \times \boxed{\prod_{\substack{1 \leq j \leq \ell \\ j \neq \bar{c}}} \Omega^3 G\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega^2 G}\end{aligned}$$

For $G = SU(n)$, $n \geq 3$, a principal $SU(n)$-bundle P is determined by

$$c_2(P) \in \mathbb{Z}^m \quad \text{and} \quad c_3(P) \in \mathbb{Z}.$$

Theorem (Cutler-S.)

Denote by $\mathcal{G}_{0,k}(M; SU(n))$ the gauge group of P with trivial $c_2(P)$ and $c_3(P) = k$. If $\mathcal{P}^1 : H^2(M; \mathbb{Z}/3) \to H^6(M; \mathbb{Z}/3)$ is trivial then

$$\mathcal{G}_{0,k}(M; SU(n)) \simeq_{(2)} \boxed{\mathcal{G}_{0,k}(M'; SU(n))} \times \boxed{\prod_{k=1}^{2n} \Omega^3 SU(n)} \times \boxed{\prod_{j=1}^{\ell} \Omega^3 SU(n)\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega^2 SU(n)}.$$

If $\mathcal{P}^1 : H^2(M; \mathbb{Z}/3) \to H^6(M; \mathbb{Z}/3)$ is non-trivial, then

$$\mathcal{G}_{0,k}(M; SU(n)) \simeq_{(2)} \boxed{\mathcal{G}_{0,k}(M''; SU(n))} \times \boxed{\prod_{k=1}^{2n} \Omega^3 SU(n)} \times \boxed{\prod_{j=1}^{\ell} \Omega^3 SU(n)\{t_j\}} \times \boxed{\prod_{i=1}^{m-1} \Omega^2 SU(n)}$$

$$\mathcal{G}_{0,k}(M; SU(n)) \simeq_{(2)} \boxed{\mathcal{G}_{0,k}(M'''; SU(n))} \times \boxed{\prod_{k=1}^{2n} \Omega^3 SU(n)} \times \boxed{\prod_{\substack{1 \leq j \leq \ell \\ j \neq \overline{c}}} \Omega^3 SU(n)\{t_j\}} \times \boxed{\prod_{i=1}^{m} \Omega^2 SU(n)}.$$

Here M', M'', M''' are 6-dim CW complexes.

Higher dimensional manifolds

M: orientable, closed, $(n-1)$-connected, $2n$-dimensional manifold

i	0	$1 \leq i \leq n-1$	n	$n+1 \leq i \leq 2n-1$	$2n$
$H_i(M)$	$\mathbb{Z}$	0	$\boxed{\mathbb{Z}^m}$	0	$\boxed{\mathbb{Z}}$

Theorem (Huang, 21)

If $n \equiv 3, 5, 6, 7 \pmod 8$ *then*

$$\Sigma M \simeq \boxed{S^{2n+1}} \vee \boxed{\bigvee_{i=1}^{m} S^{n+1}}, \quad \mathcal{G}_\alpha(M; G) \simeq \boxed{\mathcal{G}_\alpha(S^{2n})} \times \boxed{\prod_{i=1}^{m} \Omega^n G}$$

If $n \equiv 0, 1, 2, 4 \pmod 8$ *and* $m \geq 2$ *then*

$$\Sigma M \simeq \boxed{\Sigma X} \vee \boxed{\bigvee_{i=1}^{m-1} S^{n+1}}, \quad \mathcal{G}_\alpha(M; G) \simeq \boxed{\mathcal{G}_\alpha(X)} \times \boxed{\prod_{i=1}^{m-1} \Omega^n G}$$

where $X = S^n \cup e^{2n}$

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Thank you!