The Rational Homotopy Type of Homotopy Fibrations Over Connected Sums

(arXiv 2203.11064)

Notation

1. X, Y simply connected, based

$$X \rtimes Y = \frac{X \times Y}{X \times y_0}$$

⇝ If Y is co-H, then $X \rtimes Y \simeq (X \wedge Y) \vee Y$.

2. X is n-dimensional CW complex, then let $\bar{X}$ denote its $(n-1)$-skeleton.

3. N & M are n-dimensional Poincaré Duality complexes. $\exists$ homotopy cofibrations

$$S^{n-1} \xrightarrow{f} \bar{M} \longrightarrow M$$
$$S^{n-1} \xrightarrow{g} \bar{N} \longrightarrow N$$

Take $f+g : S^{n-1} \xrightarrow{\sigma} S^{n-1} \vee S^{n-1} \xrightarrow{f \vee g} \bar{M} \vee \bar{N}$.
Then connected sum $M \# N$ is the homotopy

cofiber of $f \vee g$.

$$S^{n-1} \xrightarrow{f \vee g} \bar{M} \vee \bar{N} \to M \# N.$$

§1: Inspiration

Take a homotopy fibration of Poincaré Duality complexes

$$F \longrightarrow L \xrightarrow{f} C$$

where $\dim C = n$ and $\dim L = m$.

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\pi"'] & B \# C \arrow[d, "p"] \\
F \arrow[r] & L \arrow[r, "f"'] & C
\end{tikzcd}$$

Question: to what extent does M look like a connected sum?

Theorem (Jeffrey–Selick '21) $\exists$ homotopy cofibration

$$F \ltimes \overline{B} \longrightarrow M \xrightarrow{\pi} L$$

Moreover, there is a homology isomorphism $H_k(M) \cong H_k(L) \oplus H_k(F \ltimes \overline{B})$ for $0 < k < m$

$\leadsto$ In their paper, Jeffrey–Selick give examples where $\exists$ Poincaré Duality complex X with $\overline{X} \simeq F \ltimes \overline{B}$ s.t. $M \cong X \# L$, and other situations where such an X cannot exist.

- Jeffrey–Selick, Bundles over Connected Sums (arXiv 2112.05714)
- Duan, Circle Actions and Suspension Operations on Smooth Manifolds (arXiv 2202.06225)
- Huang–Theriault, Homotopy of Manifolds Stabilised by Projective Spaces (arXiv 2204.04368)

§2: A Proposition

Def Let $A \xrightarrow{f} B \xrightarrow{j} C$ be a homotopy cofibration of simply connected spaces. The map f is called <u>inert</u> if Ωj has a right homotopy inverse.

$\rightsquigarrow \exists \ s : \Omega C \to \Omega B$ s.t. $\Omega j \circ s \simeq 1_{\Omega C}$

<u>Example</u>: $S^{n-1} \xrightarrow{g} Y \to Y \cup_g e^n$

$S^{2n-1} \xrightarrow{w} S^n \vee S^n \to S^n \times S^n$ ← Whitehead product

w is inert

<u>Theorem</u> (Theriault '21)

Let $A \xrightarrow{f} B \xrightarrow{j} C$ be a homotopy cofibration of simply connected spaces. If the map f is inert, then there exists a homotopy equivalence

$$\Omega B \simeq \Omega C \times \Omega\big((\Omega C) \ltimes A\big).$$

Take two n-dimensional Poincaré Duality complexes M & N. There are homotopy cofibrations

$$S^{n-1} \xrightarrow{f} \bar{M} \to M$$
$$S^{n-1} \xrightarrow{g} \bar{N} \to N$$

Lemma: if f is inert then $\exists$ homotopy equivalence

$$\Omega(M \# N) \simeq \Omega M \times \Omega(\Omega M \ltimes \bar{N}).$$

Proof we have a homotopy cofibration diagram

$$\begin{tikzcd}
 & \bar{N} \arrow[r, equal] \arrow[d, hook] & \bar{N} \arrow[d] \\
S^{n-1} \arrow[r, "f+g"] \arrow[d, equal] & \bar{M} \vee \bar{N} \arrow[r] \arrow[d] & M \# N \arrow[d, "h"] \\
S^{n-1} \arrow[r, "f"'] & \bar{M} \arrow[r] \arrow[u, dashed, bend right, "\Omega"'] & M \arrow[l, dashed, bend left, "\Omega"]
\end{tikzcd}$$

By homotopy commutativity of the diagram, Ωh has a right homotopy inverse.

Theriault $\Rightarrow$ $\Omega(M \# N) \simeq \Omega M \times \Omega(\Omega M \ltimes \bar{N})$.

Prop^n: take the homotopy fibration diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \# C \arrow[d, "p"] \\ F \arrow[r] & L \arrow[r, "f"] & C \end{tikzcd}$$

$m-n$ $\qquad$ m $\qquad$ n

$\bar{B}$ $(n-1)$

$F \ltimes \bar{B}$ $(n-1)$
$+ (m-n)$
$\rightsquigarrow m-1$

If Ωp has a right homotopy inverse then so does $\Omega\pi$. Moreover, if L has inertness, then there is a homotopy equivalence

$$\Omega M \simeq \Omega(X \# L)$$

where X is the 1-point compactification of $F \ltimes \bar{B}$.

Proof: consider the pullback

$$\begin{tikzcd} \Omega L \arrow[r, "\Omega f"] \arrow[dr, dotted] \arrow[ddr, bend right, "1_{\Omega L}"'] & \Omega C \arrow[dr, "s"] & \\ & \Omega M \arrow[r] \arrow[d, "\Omega\pi"] & \Omega(B \# C) \arrow[d, "\Omega p"] \\ & \Omega L \arrow[r] & \Omega C \end{tikzcd}$$

$$\Omega L \xrightarrow{\Omega f} \Omega C$$

$\exists\ \lambda : \Omega L \longrightarrow \Omega M$ s.t. $\Omega\pi \circ \lambda \simeq 1_{\Omega L}$.

Recall $\exists$ homotopy cofibration

let $X' := F \ltimes \overline{B}$

$$F \ltimes \overline{B} \longrightarrow M \xrightarrow{\pi} L$$

Theorem $\Rightarrow$ $\Omega M \simeq \Omega L \times \Omega(\Omega L \ltimes X')$.

$$\begin{tikzcd}
 & X' \arrow[r, equal] \arrow[d] & X' \arrow[d] \\
S^{m-1} \arrow[r] \arrow[d, equal] & X' \vee \overline{L} \arrow[r] \arrow[d] & X \# L \arrow[d] \\
S^{m-1} \arrow[r] & \overline{L} \arrow[r] & L
\end{tikzcd}$$

Lemma $\Rightarrow$ $\Omega(X \# L) \simeq \Omega L \times \Omega(\Omega L \ltimes X')$ $\square$

§3: Rational Homotopy

Def Let $A \xrightarrow{f} B \xrightarrow{j} C$ be a homotopy cofibration of simply connected spaces. The map f is <u>rationally inert</u> if Ωj induces a surjection in rational homology.

($\Rightarrow \Omega j$ has a right homotopy inverse)
<u>after rationalization</u>

Consider again cell attachment $S^{n-1} \xrightarrow{g} Y \to Y \cup_g e^n$

<u>Theorem</u> (Halperin–Lemaire '87) given a cell attachment as above, if $H^*(Y;\mathbb{Q})$ is generated by more than one element, and $Y \cup_g e^n$ is a Poincaré Duality complex, then the map g is rationally inert.

<u>Theorem</u> (C. '22) take the homotopy fibration diagram

$$\begin{tikzcd}
F \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \# C \arrow[d, "p"] \\
F \arrow[r] & L \arrow[r, "f"'] & C
\end{tikzcd}$$

If $H^*(\overline{C};\mathbb{Q})$ and $H^*(\overline{L};\mathbb{Q})$ are generated by more than element, then there is a rational homotopy equivalence

$$\Omega M \simeq \Omega(X \rtimes L).$$

Proof $\qquad \overline{B} = \overline{B}$

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] \\
S^{n-1} \arrow[r] \arrow[d, equal] & \bar{B} \vee \bar{C} \arrow[r] \arrow[d, shift right] & B \# C \arrow[d, "p"] \\
S^{n-1} \arrow[r] & \bar{C} \arrow[r] \arrow[u, dotted, bend right, shift right, "\Omega"'] & C \arrow[l, dotted, bend left, "\Omega"]
\end{tikzcd}$$

∎

Remark: A simply connected space Y is <u>rationally elliptic</u> if

$$\dim\left(\pi_*(Y) \otimes \mathbb{Q}\right) < \infty$$

and <u>rationally hyperbolic</u> otherwise.

B n-dim
k-conn
$n \leq 3k+1$

$$X' = F \ltimes \bar{B} \simeq (F \wedge \bar{B}) \vee \bar{B}$$

$\rightsquigarrow X'$ retracts off $\Omega L \ltimes X'$

$\Rightarrow \Omega X'$ retracts off ΩM.