On the Homotopy Groups of the Suspended Quaternionic Projective Plane

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INTRODUCTION

In this talk, I'll report

1. the computations of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ ($\forall\ r \leq 15,\ k \geq 0$) localized at 2 or 3, especially the unstable ones. (Actually, many of these groups in the stable range have been computed by *A.Liulevicius* and *J.Mukai*)

2. the applications including the classification theorems of the 3-local CW complexes

$$S^{4+k} \cup e^{8+k} \cup e^{12+k}, \quad k \geq 1$$

and some decompositions of the self smashes.

Remark: Localized at a prime $p \neq 2, 3$,

$$\Sigma\mathbb{H}P^2 \simeq S^5 \vee S^9, \quad \pi_*(\Sigma^k \mathbb{H}P^2) = \pi_*(S^{4+k} \vee S^{8+k}), \ k \geq 1$$

Main tools

TOOL 1: the fibre sequence

$$J(X,A) \to X \cup CA \xrightarrow{pinch} \Sigma A \quad \text{for a CW pair } (X,A),$$

where $J(X,A)$ is the **relative James construction** defined by $B.Gray$.

TOOL 2: $J.Mukai$'s works on the order of the Whitehead products

$$[1_{S^n}, \alpha] \in \pi_{n+m-1}(S^n),\ \alpha \in \pi_{n+m}(S^n) \text{ and } n \geq m+2$$

TOOL 3: The chasing method, i.e. compare the generators of $\pi_m(\Sigma^k \mathbb{H}P^2)$ with the generators of $\pi_{m+t}(\Sigma^{k+t}\mathbb{H}P^2)$ or $\pi^S_{m-k}(\mathbb{H}P^2)$.

The Relative James Construction

★ Such as the famous (absolute) James construction $J(X)$ for a CW complex X,

$$\begin{array}{ccccccc} J_0(X) & \subseteq & J_1(X) & \subseteq & J_2(X) & \subseteq & \cdots, \\ \| & & \| & & & & \\ * & & X & & & & \end{array} \qquad J(X) = \bigcup_{n\geq 0} J_n(X),$$

satisfying with $J_n(X)/J_{n-1}(X) = X^{\wedge n}$, $\qquad \Sigma J(X) \simeq \Sigma \bigvee_{n\geq 1} X^{\wedge n}$.

★ Defined and studied by *B.Gray* , there also exists the relative James construction $J(X,A)$ for a CW pair (X,A),

$$\begin{array}{ccccccc} J_0(X,A) & \subseteq & J_1(X,A) & \subseteq & J_2(X,A) & \subseteq & \cdots, \\ \| & & \| & & & & \\ * & & X & & & & \end{array} \qquad J(X,A) = \bigcup_{n\geq 0} J_n(X,A),$$

satisfying with $J_n(X,A)/J_{n-1}(X,A) = X\wedge A^{\wedge(n-1)}$, $\quad \Sigma J(X,A) \simeq \Sigma \bigvee_{n\geq 0} X\wedge A^{\wedge n}$.

3

Properties of the relative James construction

Proposition (B.Gray ,[1]) : For a CW pair (X, A), let $i : A \hookrightarrow X$ be the inclusion,

(1) there exists a fibre sequence $J(X, A) \to X \cup_i CA \xrightarrow{pinch} \Sigma A$

(2) $\widetilde{H}_*(J(X, A)) \approx \widetilde{H}_*(X) \otimes H_*(\Omega\Sigma A)$, coefficients in a **field**.

(3) $J_2(X, A) = X \cup_{[1_X, i]} C(X \wedge A')$, if $X = \Sigma X'$ and $A = \Sigma A'$.
($[1_X, i]$:Whitehead product)

Properties of the relative James construction

(4) $J_n(X,A)/J_{n-1}(X,A) = X \wedge A^{\wedge(n-1)}$, $\Sigma J(X,A) \simeq \Sigma \bigvee_{n\geq 0} X \wedge A^{\wedge n}$,

they are natural for pairs.

(5) there are the n-th relative James-Hopf invariants

$$\mathcal{H}_n : J(X,A) \to J(X \wedge A^{\wedge(n-1)}),$$ which are natural for pairs.

CLAIM: In next part, by abuse of notations , for any map $f : U \to V$, the symbol M_U denotes the mapping cylinder of f, to indicate $M_U \simeq U$.

5

Immediately, by checking homology,

The Skeletons of J

Let Y be a CW complex and $2 \leq dim(Y) = m' \leq m$.

For $Z = Y \cup_f e^{m+1}$, we have

$$sk_r(J(M_Y, S^m)) = J_t(M_Y, S^m)\ , r = m(t-1) + m' - 1.$$

The Fibre of $\Sigma^k\mathbb{H}P^2 \xrightarrow{pinch} S^{8+k}$

From now on, F_k always denotes the fibre of $\Sigma^k\mathbb{H}P^2 \xrightarrow{pinch} S^{8+k}$.

Diagram with rows fibre sequences $\implies$ $F_k \simeq J(M_{S^{4+k}}, S^{7+k})$.

$$\begin{tikzcd}
F_k \arrow[rr] \arrow[d] & & \Sigma^k\mathbb{H}P^2 \arrow[r, "p_k", "pinch"'] \arrow[d, "\simeq"] & S^{8+k} \arrow[d, "id"'] \\
J(M_{S^{4+k}}, S^{7+k}) \arrow[r] & & M_{S^{4+k}} \cup_i CS^{7+k} \arrow[r, "p'_k", "pinch"'] & S^{8+k}
\end{tikzcd}$$

This provides more homotopy information of F_k .
(than the Serre $s.s.$ for $\Omega S^{8+k} \longrightarrow F_k \longrightarrow \Sigma^k\mathbb{H}P^2$)

By $B.\,Gray's\ \ J_2(X,A)$, we have

The Skeletons of F_k

Lemma : Suppose $k \geq 1$, then

after localization at 2,

$$sk_{18+3k}(F_k) = S^{4+k} \cup_{f_k} e^{11+2k},$$

where $f_k = [\iota_{4+k}, \nu_{4+k}]$ is the Whitehead product;

and after localization at 3,

$$sk_{18+3k}(F_k) = S^{4+k} \cup_{g_k} e^{11+2k},$$

where $g_k = [\iota_{4+k}, \alpha_1(4+k)]$ is the Whitehead product and $\alpha_1(4+k)$ is the generator of $\pi_{7+k}(S^{4+k}) \approx \mathbb{Z}/3$.

8

Mukai's results on the orders of the Whitehead products,

$$\Downarrow$$

The Skeletons of F_k

Lemma :

After localization at 2 or 3,

$$sk_{20}(F_1) = S^5 \vee S^{13} \ , sk_{26}(F_3) = S^7 \vee S^{17};$$

after localization at 2,

$$sk_{23}(F_2) = S^6 \cup_{2\overline{\nu}_6} e^{15}, \ sk_{29}(F_4) = S^8 \cup_{f_4} e^{19},$$

$$sk_{32}(F_5) = S^9 \cup_{\overline{\nu}_9\nu_{17}} e^{21}, \ sk_{35}(F_6) = S^{10} \cup_{\Delta(\nu_{21})} e^{23},$$

where $f_4 = \nu_8\sigma_{11} - 2\sigma_8\nu_{15}$.

9

The Chasing Method

Diagram with the fibre sequences as rows ,

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d] & F_k \arrow[r, "i_k"] \arrow[d, "\varphi_{k+t}"] & \Sigma^k \mathbb{H}P^2 \arrow[r, "p_k"] \arrow[d, "w_{k+t}=\Omega^t\Sigma^t"] & S^{8+k} \arrow[d, "\theta_{k+t}=\Omega^t\Sigma^t"] \\
\cdots \arrow[r] & \Omega^{t+1} S^{8+k+t} \arrow[r, "\Omega^t\partial_{k+t}"] & \Omega^t F_{k+t} \arrow[r, "\Omega^t i_{k+t}"] & \Omega^t\Sigma^{k+t}\mathbb{H}P^2 \arrow[r, "\Omega^t p_{k+t}"] & \Omega^t S^{8+k+t}
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{m+1}(S^{8+k}) \arrow[r, "\partial_{k*_{m+1}}"] \arrow[d, "\theta_{k+t_*}"] & \pi_m(F_k) \arrow[r, "i_{k*}"] \arrow[d, "\varphi_{k+t_*}"] & \pi_m(\Sigma^k \mathbb{H}P^2) \arrow[r, "p_{k*}"] \arrow[d, "w_{k+t_*}"] & \pi_m(S^{8+k}) \arrow[d, "\theta_{k+t_*}"] \\
\cdots \arrow[r] & \pi_{m+1+t}(S^{8+k+t}) \arrow[r, "\partial_{k+t*_{m+1}}"] & \pi_{m+t}(F_{k+t}) \arrow[r, "i_{k+t_*}"] & \pi_{m+t}(\Sigma^{k+t} \mathbb{H}P^2) \arrow[r, "p_{k+t_*}"] & \pi_{m+t}(S^{8+k+t})
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
0 \arrow[r] & coker(\partial_{k*_{m+1}}) \arrow[r] \arrow[d] & \pi_m(\Sigma^k \mathbb{H}P^2) \arrow[r] \arrow[d] & Ker(\partial_{k*_m}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & coker(\partial_{k+t*_{m+1+t}}) \arrow[r] & \pi_{m+t}(\Sigma^{k+t} \mathbb{H}P^2) \arrow[r] & Ker(\partial_{k+t*_{m+t}}) \arrow[r] & 0
\end{tikzcd}$$

The Chasing Method

THE CHASING METHOD IS:

$$\mathrm{coker}(\partial_{k_{*m+1}})\ ,\ \mathrm{Ker}(\partial_{k_{*m}})$$

$$\Downarrow$$

an extension problem

$$0 \to coker(\partial_{k_{*m+1}}) \to \pi_m(\Sigma^k \mathbb{H}P^2) \to Ker(\partial_{k_{*m}}) \to 0;$$

Solve it, get solutions;

To exclude the extra solutions,

$$\text{the solutions} \quad \text{“VS”} \quad \text{known } \pi_{m+t}(\Sigma^{k+t}\mathbb{H}P^2) \text{ or } \pi^S_{m-k}(\mathbb{H}P^2)$$

by the two diagrams (yellow) of the precious page.

12

My methods to get $coker(\partial_{k*})$ and $Ker(\partial_{k*})$ are:

1. three homotopy-commutative diagrams (*in next pages*) ;
2. *H.Toda*'s works on the composition relations of the generators of $\pi_*(S^n)$.

Suppose $\mathbb{H}P^2 = S^4 \cup_{\mathbf{a}} e^8$.

Cofibre sequence $S^{4+k} \longrightarrow \underline{\Sigma^k \mathbb{H}P^2 \xrightarrow{p_k} S^{8+k} \xrightarrow{-\Sigma^k \mathbf{a}} S^{5+k}}$

$\Downarrow$

Diagram–1 to get $coker(\partial_{k*})$ and $Ker(\partial_{k*})$

Diagram with rows fibre sequences

$$\begin{tikzcd}
\Omega S^{8+k} \arrow[r, "\partial_k"] \arrow[d, "id"] & F_k \arrow[r] \arrow[d] & \Sigma^k \mathbb{H}P^2 \arrow[r] \arrow[d] & S^{8+k} \arrow[d, "id"] & \\
\Omega S^{8+k} \arrow[r, "\Omega\Sigma^k \mathbf{a}"] & \Omega S^{5+k} \arrow[r] & J(\Sigma^k \mathbb{H}P^2, S^{4+k}) \arrow[r] & S^{8+k} \arrow[r, "-\Sigma^k \mathbf{a}"] & S^{5+k}.
\end{tikzcd}$$

Diagram–2 to get $coker(\partial_{k*})$ and $Ker(\partial_{k*})$

$$\begin{tikzcd} S^{4+k} \arrow[d, "\subseteq"] \arrow[dr, "\subseteq"] & \\ J(S^{4+k}) \arrow[r, "\partial_k"] & J(M_{S^{4+k}}, S^{8+k}) \end{tikzcd}$$

Diagram–3 to get $coker(\partial_{k*})$ and $Ker(\partial_{k*})$

($\mathcal{H}_2$: the 2nd relative James-Hopf invariant)

$$\begin{tikzcd}
S^{4+k} \arrow[r, "\subseteq"] \arrow[d, "\subseteq"] & J_2(M_{S^{4+k}}, S^{8+k}) \arrow[r, "pinch"] \arrow[d, "\subseteq"] & S^{12+2k} \arrow[d, "\subseteq"] \\
J(S^{4+k}) \arrow[r, "\partial_k"] & J(M_{S^{4+k}}, S^{8+k}) \arrow[r, "\mathcal{H}_2"] & J(S^{12+2k})
\end{tikzcd}$$

A rough partition

My computations , of $\pi_{r+k}(\Sigma^k \mathbb{H}P^2)$ ($\forall\ r \leq 15,\ k \geq 0$), could roughly part into 4 cases. The later a case we state, the more complicated it will be.

The 4 cases to compute $\pi_m(\Sigma^k \mathbb{H}P^2)$

$$\pi_{m+1}(S^{8+k}) \xrightarrow{\partial_{k*m+1}} \pi_m(F_k) \longrightarrow \pi_m(\Sigma^k \mathbb{H}P^2) \longrightarrow \pi_m(S^{8+k}) \xrightarrow{\partial_{k*m}} \pi_{m-1}(F_k)$$

Case 1: The stable groups.

Case 2: Only need the 1-cell skeleton of F_k.

with 4 subcases,

Subcase 2.1: There're many trivial groups in the above sequence ;

Subcase 2.2: coker(∂_{k*m+1})=0 or Ker(∂_{k*m+1}) is free;

Subcase 2.3: coker$(\partial_{k*m+1}) \neq 0$ and Ker(∂_{k*m+1}) is not free, but they are both finite;

Subcase 2.4: coker$(\partial_{k*m+1}) \neq 0$ and Ker(∂_{k*m+1}) is not free, and one of them is infinite.

The 4 cases

$$\pi_{m+1}(S^{8+k}) \xrightarrow{\partial_{k*m+1}} \pi_m(F_k) \longrightarrow \pi_m(\Sigma^k \mathbb{H}P^2) \longrightarrow \pi_m(S^{8+k}) \xrightarrow{\partial_{k*m}} \pi_{m-1}(F_k)$$

Case 3: Need the 2-cell skeleton of F_k.

Case 4: $\pi_{11}(\Sigma^2\mathbb{H}P^2)$ localized at 2.

Example Computation of Subcase 2.4

After computing the coker and Ker,

$\Longrightarrow$

$$0 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \to \pi_{15}(\Sigma^4\mathbb{H}P^2) \to \mathbb{Z}/4 \to 0\ ,$$

the solutions: $\pi_{15}(\Sigma^4\mathbb{H}P^2) = \mathbb{Z}_{(2)} \oplus A$,

where $A = \mathbb{Z}/16,\ \mathbb{Z}/16 \oplus \mathbb{Z}/2,\ \mathbb{Z}/32,\ \mathbb{Z}/32 \oplus \mathbb{Z}/2,\ \mathbb{Z}/64$ *or* $\mathbb{Z}/16 \oplus \mathbb{Z}/4$.

By the chasing method,
(to exclude the extra five solutions)

$\Longrightarrow$

$$\pi_{15}(\Sigma^4\mathbb{H}P^2) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/16 \oplus \mathbb{Z}/4$$

Compute $\pi_{17}(\Sigma^2\mathbb{H}P^2)$ localized at 2— Example Computation of Case 3

$$\pi_{18}(S^{10}) \xrightarrow{\partial_{2*18}} \pi_{17}(F_2) \xrightarrow{i_{2*}} \pi_{17}(\Sigma^2\mathbb{H}P^2) \xrightarrow{p_{2*}} \pi_{17}(S^{10}) \xrightarrow{\partial_{2*17}} \pi_{16}(F_2)$$

We need $\mathrm{cok}(\partial_{2*18})$ and $\mathrm{Ker}(\partial_{2*17})$. Firstly, we observe the homotopy-commutative diagram, where H_2' and H_2 are the 2nd relative James-Hopf invariants ,

$$\begin{tikzcd}
J(S^9) \arrow[r, "\partial_2"] \arrow[ddd, "H_2'"] & J(M_{S^6}, S^9) \arrow[ddd, "H_2"] & J_2(M_{S^6}, S^9) \arrow[l, "\supseteq"'] \arrow[r, "pinch\ M_{S^6}"] \arrow[d, "\Omega\Sigma"'] & S^{15} \arrow[d, "\Omega\Sigma"] \\
 & & \Omega(S^7 \vee S^{16}) \arrow[r] \arrow[dd, "\Omega\, pinch"] & \Omega S^{16} \arrow[dd, "id"] \\
 & & & \\
J(S^{18}) \arrow[r] & J(S^{15}) & \Omega S^{16} \arrow[l, "\simeq"'] \arrow[r, "id"] & \Omega S^{16}
\end{tikzcd}$$

it induces the commutative diagram (after several identifications of the isomorphism groups),

$$\begin{tikzcd}
\pi_{18}(S^{10}) \arrow[r, "\partial_{2*18}"] \arrow[d, "H'_{2*}=0"] & \pi_{17}(F_2) \arrow[r, "p'_{2*}"] \arrow[d, "H_{2*}"] & \pi_{17}(S^{15}) \arrow[d, "\approx"] \\
\pi_{18}(S^{19}) \arrow[r] & \pi_{18}(S^{16}) \arrow[r, "id"] & \pi_{18}(S^{16})
\end{tikzcd}$$

21

Compute $\pi_{17}(\Sigma^2\mathbb{H}P^2)$ localized at 2— Example Computation of Case 3

Thus, $H_{2*} \circ \partial_{2*_{18}} = 0$, this means the following the composition is zero,

$$\pi_{18}(S^{10}) \xrightarrow{\partial_{2*_{18}}} \pi_{17}(F_2) \xrightarrow{proj.} \pi_{17}(S^{15}) \xrightarrow{\approx} \pi_{18}(S^{16})$$

So, for any $u \in \pi_{18}(S^{10}), \partial_{2*_{18}}(u) = x_1 j_2\xi_6 + x_2 j_2\overline{\nu}_6\nu_{14} + x_3\widehat{j_2\eta^2_{15}}$ implies $x_3 \equiv 0 \ (mod\ 2)$. Hence, $\mathrm{Im}(\partial_{2*_{18}}) \subseteq \mathbb{Z}/8\{j_2\xi_6\} \oplus \mathbb{Z}/2\{j_2\overline{\nu}_6\nu_{14}\}$. Then we consider the homotopy-commutative diagram,

$$\begin{tikzcd}
\Omega S^{10} \arrow[r, "\partial_2"] \arrow[d, "id"] & F_2 \arrow[d, "\phi_2"] & S^6 \arrow[l, "j_2"'] \arrow[dl, "\Omega\Sigma"] \\
\Omega S^{10} \arrow[r, "\Omega\nu_7"] & \Omega S^7 &
\end{tikzcd}$$

$$\begin{tikzcd}
\pi_{18}(S^{10}) \arrow[r, "\partial_{2*_{18}}"] \arrow[d, "id"] & \pi_{17}(F_2)/\langle\widehat{j_2\eta^2_{15}}\rangle \arrow[d, "\underline{\phi}_{2*}"'] & \pi_{17}(S^6) \arrow[l, "\underline{j}_{2*}"'] \arrow[dl, "\Sigma"] \\
\pi_{18}(S^{10}) \arrow[r, "\nu_{7*}"] & \pi_{18}(S^7) &
\end{tikzcd}$$

Compute $\pi_{17}(\Sigma^2\mathbb{H}P^2)$ localized at 2— Example Computation of Case 3

Since $\pi_{18}(S^{10}) = \mathbb{Z}/2\{\overline{\nu}_{10}\}\oplus\mathbb{Z}/2\{\varepsilon_{10}\}$, $\pi_{17}(F_2)/\langle\widehat{j_2\eta^2_{15}}\rangle = \mathbb{Z}/8\{j_2\xi_6\}\oplus\mathbb{Z}/2\{j_2\overline{\nu}_6\nu_{14}\}$, $\pi_{17}(S^6) = \mathbb{Z}/8\{\xi_6\}\oplus\mathbb{Z}/4\{\overline{\nu}_6\nu_{14}\}$, $\pi_{18}(S^7) = \mathbb{Z}/8\{\xi_7\}\oplus\mathbb{Z}/2\{\overline{\nu}_7\nu_{15}\}$, so $\underline{\phi}_{2_*}$ is an isomorphism. According to *Page* 70 *of* [1], we have $\nu_{7*} = 0$. Thus, $\partial_{2_{*18}} = 0$, successively, $\mathrm{cok}(\partial_{2_{*18}}) \approx \pi_{17}(F_2)/\langle\widehat{j_2\eta^2_{15}}\rangle = \mathbb{Z}/8\{j_2\xi_6\}\oplus\mathbb{Z}/2\{j_2\overline{\nu}_6\nu_{14}\}$.

Next, we observe the commutative diagram ,

$$\begin{tikzcd}
 & \mathbb{Z}/8\{\nu_6\sigma_9\}\oplus\mathbb{Z}/2\{\eta_6\mu_7\} \arrow[d, equal] \\
\pi_{16}(S^9) \arrow[r, "\nu_{6*}"] \arrow[d, "\approx"'] & \pi_{16}(S^6) \arrow[d, "into"] \\
\pi_{17}(S^{10}) \arrow[r, "\partial_{2_{*17}}"] \arrow[d, equal] & \pi_{16}(F_2) \arrow[d, equal] \\
\mathbb{Z}/16\{\sigma_{10}\} & \mathbb{Z}/8\{j_2\nu_6\sigma_9\}\oplus\mathbb{Z}/2\{j_2\eta_6\mu_7\}\oplus\mathbb{Z}/2\{\widehat{j_2\eta_{15}}\}
\end{tikzcd}$$

Hence, $\mathrm{Ker}(\partial_{2_{*17}}) = \mathbb{Z}/2\{8\sigma_{10}\}$. Therefore, we get the exact sequence,

$$0\to\mathbb{Z}/8\{j_2\xi_6\}\oplus\mathbb{Z}/2\{j_2\overline{\nu}_6\nu_{14}\}\longrightarrow\pi_{17}(\Sigma^2\mathbb{H}P^2)\longrightarrow\mathbb{Z}/2\{8\sigma_{10}\}\to 0$$

solving, chasing

$$\Downarrow$$

$$\pi_{17}(\Sigma^2\mathbb{H}P^2)\approx\mathbb{Z}/16\oplus\mathbb{Z}/2$$

23

Case 4

For Case 4, all methods we used in Case 1∼3 do not work. The chasing method has its troubles for $\pi_{11}(\Sigma^2\mathbb{H}P^2)$, due to $\pi_{4+i}(S^i) \neq 0$ and $\pi_{5+i}(S^i) \neq 0$ for some small i, while in the stable range they are 0.

Previous methods

$\Longrightarrow$

$\pi_{11}(\Sigma^2\mathbb{H}P^2) = \mathbb{Z}_{(2)}$ *or* $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$

Compute $\pi_{11}(\Sigma^2\mathbb{H}P^2)$ localized at 2

Let $U = J(\Sigma\mathbb{H}P^2)$, $V = J(\Sigma\mathbb{H}P^2)/\Sigma\mathbb{H}P^2$.

Cofibre sequence $\quad \Sigma\mathbb{H}P^2 \to U \to V \xrightarrow{p} \Sigma^2\mathbb{H}P^2$

$\Longrightarrow$

Fibre sequence $\quad J(U, \Sigma\mathbb{H}P^2) \to V \xrightarrow{p} \Sigma^2\mathbb{H}P^2$

$\Longrightarrow$

An injection $\quad \pi_{11}(\Sigma^2\mathbb{H}P^2) \hookrightarrow \pi_{10}(J(U, \Sigma\mathbb{H}P^2)) \approx \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(2)}$,

$\Longrightarrow$

$$\pi_{11}(\Sigma^2\mathbb{H}P^2) = \mathbb{Z}_{(2)}.$$

My results (localized at 2)

$$\pi_{7+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 1, \\ \mathbb{Z}/4, & k = 0 \end{cases}$$

$$\pi_{8+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(2)}, & k \geq 1, \\ \mathbb{Z}/2, & k = 0 \end{cases}$$

$$\pi_{9+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2, & k \geq 3 \ or \ k = 0, \\ \mathbb{Z}_{(2)}\ , & k = 2, \\ 0, & k = 1 \end{cases}$$

$$\pi_{10+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2, & k \geq 1, \\ 0, & k = 0 \end{cases}$$

My results (localized at 2)

$$\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/16 \oplus \mathbb{Z}/4, & k \geq 5 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}_{(2)}, & k = 4 \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4, & k = 3 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/4, & k = 2 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/4, & k = 1 \\ \mathbb{Z}_{(2)} \ , & k = 0 \end{cases}$$

$$\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} (\mathbb{Z}/2)^2, & k \geq 6 \text{ or } k = 0 \\ (\mathbb{Z}/2)^3, & k = 5 \text{ or } 3 \\ (\mathbb{Z}/2)^4, & k = 4, \\ \mathbb{Z}/2, & k = 2, \\ \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2, & k = 1 \end{cases}$$

My results (localized at 2)

$$\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} (\mathbb{Z}/2)^2, & k \geq 7 \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}_{(2)}, & k = 6 \text{ or } 2 \\ (\mathbb{Z}/2)^3, & k = 5,\ 3,\ 1 \text{ or } 0 \\ (\mathbb{Z}/2)^4, & k = 4 \end{cases}$$

$$\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/2, & k \geq 8 \text{ or } k = 4 \\ (\mathbb{Z}/2)^2, & k = 7,\ 6\ ,5,\ 2 \text{ or } 1 \\ \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}, & k = 3, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2, & k = 0 \end{cases}$$

My results (localized at 2)

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/128, & k \geq 7 \text{ } \mathit{but} \text{ } k \neq 8 \\ \mathbb{Z}/128 \oplus \mathbb{Z}_{(2)}, & k = 8 \\ \mathbb{Z}/64, & k = 6, \\ \mathbb{Z}/32, & k = 5, \\ \mathbb{Z}/32 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}_{(2)}, & k = 4, \\ \mathbb{Z}/16 \oplus (\mathbb{Z}/2)^2, & k = 3 \\ \mathbb{Z}/16 \oplus \mathbb{Z}/2, & k = 2, \\ \mathbb{Z}/16 \oplus \mathbb{Z}/8, & k = 1, \\ (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/4, & k = 0 \end{cases}$$

My results (localized at 3)

$$\pi_{7+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 1, \\ \mathbb{Z}/3, & k = 0 \end{cases}, \quad \pi_{8+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(3)}, & k \geq 2, \\ 0, & k = 0, 1 \end{cases}$$

$$\pi_{9+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}_{(3)}, & k = 2, \\ 0, & else \end{cases}, \quad \pi_{10+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 1, \\ \mathbb{Z}/3, & k = 0 \end{cases}$$

My results (localized at 3)

$$\pi_{11+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/9, & k \geq 1 \text{ and } k \neq 4, \\ \mathbb{Z}/9 \oplus \mathbb{Z}_{(3)}, & k = 4 \\ \mathbb{Z}/3 \oplus \mathbb{Z}_{(3)}, & k = 0 \end{cases}$$

$$\pi_{12+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 0 \text{ but } k \neq 1, \\ \mathbb{Z}_{(3)}, & k = 1 \end{cases}$$

$$\pi_{13+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} 0, & k \geq 0 \text{ and } k \neq 2,\ 6 \\ \mathbb{Z}_{(3)}, & k = 2 \text{ or } 6 \end{cases}$$

My results (localized at 3)

$$\pi_{14+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/3, & k \geq 5 \text{ or } k = 4,\ 2 \\ \mathbb{Z}_{(3)} \oplus \mathbb{Z}/3, & k = 3 \\ \mathbb{Z}/9, & k = 1 \\ \mathbb{Z}/3 \oplus \mathbb{Z}/3, & k = 0 \end{cases}$$

$$\pi_{15+k}(\Sigma^k \mathbb{H}P^2) \approx \begin{cases} \mathbb{Z}/27, & k \geq 9 \text{ or } k = 7, 6, 5 \text{ or } 3 \\ \mathbb{Z}/27 \oplus \mathbb{Z}_{(3)}, & k = 4 \text{ or } 8 \\ \mathbb{Z}/9, & k = 2 \\ \mathbb{Z}/9 \oplus \mathbb{Z}/3, & k = 1 \\ \mathbb{Z}/3, & k = 0 \end{cases}$$

Now, let's appreciate the applications of these groups.

Applications

Localized at 3:

$\pi_{11+k}(\Sigma^k \mathbb{H}P^2)$ $(k \geq 0)$ and the Stenoord module structures,

$\Longrightarrow$

Two classification theorems of CW complexes $S^{4+k} \cup e^{8+k} \cup e^{12+k}$.

In next part, set $\pi_{12}(S^5) = \mathbb{Z}/3\{a\}, \pi_8(S^5) = \mathbb{Z}/3\{b\}$ and i, j are the inclusions.

Applications— Classification Theorem 1

Theorem After localization at 3, suppose $A = S^5 \cup_a e^{13}$, $c_k = \Sigma^{k-1}(ia+jb)$, then ,up to homotopy,

(**i**) for $k \geqslant 1$ and $k \neq 4$, the CW complexes of type $S^{4+k} \cup e^{8+k} \cup e^{12+k}$ can be classified as

$\Sigma^k \mathbb{H}P^3, \quad \Sigma^k \mathbb{H}P^2 \cup_{3\Sigma^k h} e^{12+k}, \quad \Sigma^k \mathbb{H}P^2 \vee S^{12+k},$

$\Sigma^{k-1} A \vee S^{8+k}, S^{4+k} \vee \Sigma^{4+k}\mathbb{H}P^2, \ (S^{4+k} \vee S^{8+k}) \cup_{c_k} e^{12+k}$ and $S^{4+k} \vee S^{8+k} \vee S^{12+k}$;

(**ii**) the CW complexes of type $S^8 \cup e^{12} \cup e^{16}$ can be classified as

$\Sigma^4 \mathbb{H}P^3, \quad \Sigma^4 \mathbb{H}P^2 \cup_{3\Sigma^4 h} e^{16}, \quad \Sigma^4 \mathbb{H}P^2 \vee S^{16},$
$C_{3^r u}$, (r runs over $\mathbb{N}$), $C_{3^r u + \Sigma^4 h}$, (r runs over $\mathbb{N}$), $C_{3^r u + 3\Sigma^4 h}$, (r runs over $\mathbb{N}$),
$\Sigma^3 A \vee S^{12}, S^8 \vee \Sigma^8 \mathbb{H}P^2, \ (S^8 \vee S^{12}) \cup_{c_4} e^{16}$ and $S^8 \vee S^{12} \vee S^{16}$.

Applications— Classification Theorem 2

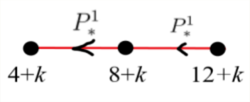
Theorem After localization at 3, for a CW complex X of type

$$S^{4+k} \cup e^{8+k} \cup e^{12+k}\,(k \geqslant 1 \text{ and } k \neq 4)\ ,$$

if $\widetilde{H}_*(X)$ has nontrivial Steenord operations P^1_* of dimension $8+k$ and $12+k$, then

$$X \simeq \Sigma^k \mathbb{H}P^3\ .$$

The Steenord module structure of these X in this picture $\implies$ $X \simeq \Sigma^k \mathbb{H}P^3$.



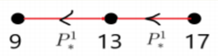
After localization at 3,

$\widetilde{H}_*(\mathbb{H}P^2)$=$\mathbb{Z}/3\{x,y\} := V$, $|x| = 4, |y| = 8$.

$\widetilde{H}_*(\mathbb{H}P^2 \wedge \mathbb{H}P^2) = V \otimes V = \mathbb{Z}/3\{xx, xy, yx, yy\}$. Choose $u = \frac{1+(12)}{2}$, $v = \frac{1-(12)}{2} \in \mathbb{Z}_{(3)}[S_2]$. They decide two self maps of $\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^3$, $1 = u + v$ is an orthogonal decomposition of $1 \in \mathbb{Z}_{(3)}[S_2]$. By *Selick and Wu's Formula*,we have

$$\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2 \simeq \mathrm{hocolim}_u(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2) \vee \mathrm{hocolim}_v(\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2).$$

By checking Stenoord module structures of these two homotopy colimits, and by our Classification Theorem 2,



Theorem

After localization at 3,

$$\Sigma\mathbb{H}P^2 \wedge \mathbb{H}P^2 \simeq S^{13} \vee \Sigma^5\mathbb{H}P^3,$$

Similarly ,

Theorem

After localization at 3,

$$\Sigma\mathbb{H}P^3 \wedge \mathbb{H}P^3 \simeq \Sigma^9\mathbb{H}P^3 \vee Y,$$

where Y is a 6-cell complex with $sk_{13}(Y) = \Sigma^5\mathbb{H}P^2$.

By the splitting of $\Sigma(\mathbb{H}P^2)^{\wedge 2}$, we get an interesting splitting,

Corollary

After localization at 3,

$$\Sigma(\mathbb{H}P^2)^{\wedge 3} \simeq \Sigma^{13}\mathbb{H}P^2 \bigvee \Sigma^5\mathbb{H}P^3 \wedge \mathbb{H}P^2,$$

$\Sigma^5\mathbb{H}P^3 \wedge \mathbb{H}P^2$ is undecomposable.

37

Thank You!