Path homology and join of digraphs

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Contents

1 Path homology of digraphs

1.1 Paths in a finite set

Let V be a finite set. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, ..., i_p$ of $p+1$ vertices of V.

Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Any element of Λ_p is called a *p-path*.

An elementary p-path $i_0, ..., i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \qquad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle, \qquad \text{etc}$$

Define also an elementary (-1)-path as the unity e of $\mathbb{K}$ so that

$$\Lambda_{-1} = \langle e \rangle = \mathbb{K}.$$

Any p-path u can be written in a form

$$u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p},$$

where $u^{i_0 i_1 \dots i_p} \in \mathbb{K}$.

Definition. Define for any $p \geq 0$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \dots i_p} = \sum_{q=0}^{p} (-1)^q \, e_{i_0 \dots \widehat{i_q} \dots i_p},$$

where $\widehat{\ }$ means omission of the index.

For example,

$$\partial e_i = e, \qquad \partial e_{ij} = e_j - e_i, \ \partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}, \quad \text{etc.}$$

Lemma 1.1 $\partial^2 = 0$.

Proof. Indeed, for any $p \geq 1$ we have

$$\begin{aligned}\partial^2 e_{i_0 \dots i_p} &= \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \dots \widehat{i_q} \dots i_p} = \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p} \right) \\ &= \sum_{0 \leq r < q \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_r} \dots \widehat{i_q} \dots i_p} - \sum_{0 \leq q < r \leq p} (-1)^{q+r} \, e_{i_0 \dots \widehat{i_q} \dots \widehat{i_r} \dots i_p}.\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0...i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$. ■

Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_{-1} \overset{\partial}{\leftarrow} \Lambda_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \dots$$

Definition. An elementary p-path $e_{i_0...i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, ..., p-1$, and irregular otherwise.

Let I_p be the subspace of Λ_p spanned by irregular $e_{i_0...i_p}$. We claim that $\partial I_p \subset I_{p-1}$. Indeed, if $e_{i_0...i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\begin{aligned}\partial e_{i_0...i_p} &= e_{i_1...i_p} - e_{i_0 i_2...i_p} + ... \\ &\quad + (-1)^k e_{i_0...i_{k-1} i_{k+1} i_{k+2}...i_p} + (-1)^{k+1} e_{i_0...i_{k-1} i_k i_{k+2}...i_p} \\ &\quad + ... + (-1)^p e_{i_0...i_{p-1}}.\end{aligned} \tag{1.1}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.1) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0...i_p} \in I_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / I_p$, and we obtain the chain complex $\mathcal{R}_*(V)$:

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \dots$$

By setting all irregular p-paths to be equal to 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$.

1.2 Chain complex and path homology of a digraph

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and a set $E \subset \{V \times V \setminus \text{diag}\}$ of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0...i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, ..., p-1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p(G)$ be $\mathbb{K}$-linear space spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \langle e_{i_0...i_p} : i_0...i_p \text{ is allowed} \rangle.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{c}{\bullet}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is not allowed.

Definition. A p-path u is called ∂-*invariant* if $u \in \mathcal{A}_p$ and $\partial u \in \mathcal{A}_{p-1}$.

The space of ∂-invariant paths is denoted by Ω_p:

$$\boxed{\Omega_p = \{u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1}\}}.$$

Important: $\partial\Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Hence, we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$0 \leftarrow \Omega_{-1} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

Note that $\Omega_{-1} = \mathbb{K}$, $\Omega_0 = \mathcal{A}_0 = \langle e_i, i \in V\rangle$ and $\Omega_1 = \mathcal{A}_1 = \langle e_{ij}, i \to j\rangle$, while in general $\Omega_p \subset \mathcal{A}_p$.

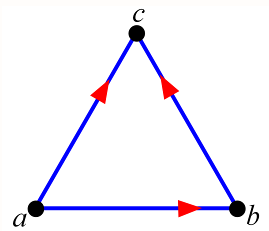
1.3 Examples of ∂-invariant paths

A *triangle* is a sequence of three vertices a, b, c such that

$$a \to b \to c, \;\; a \to c.$$

It determines 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1.$$



A *square* is a sequence of four vertices a, b, b', c such that

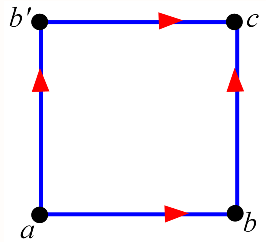
$$a \to b, \; b \to c, \; a \to b', \; b' \to c.$$

It determines a 2-path

$$u = e_{abc} - e_{ab'c} \in \Omega_2$$

because $u \in \mathcal{A}_2$ and

$$\begin{aligned}\partial u &= \left(e_{bc} - \underline{e_{ac}} + e_{ab}\right) - \left(e_{b'c} - \underline{e_{ac}} + e_{ab'}\right) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1\end{aligned}$$

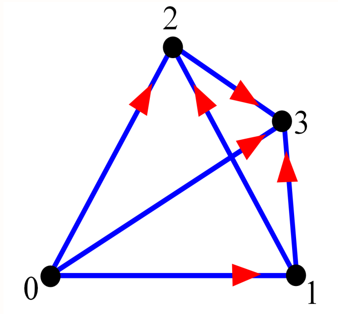


In general, Ω_2 has a basis that consists of triangles and squares and double arrows e_{aba}.

A p-*simplex* (or p-clique) is a sequence of $p+1$ vertices, say, $0,1,...,p$, such that

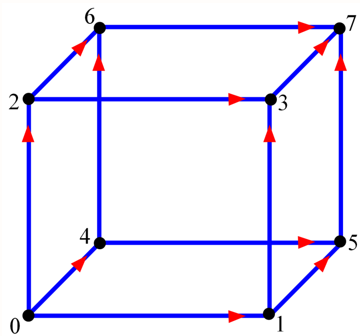
$$i \to j \Leftrightarrow i < j.$$

It determines a p-path $e_{01\dots p} \in \Omega_p$.



1-simplex is $\bullet \to \bullet$,   2-simplex is a triangle.
Here is a 3-simplex:

A 3-*cube* is a sequence of 8 vertices $0,1,2,3,4,5,6,7$, connected by arrows as here:
It determines a ∂-invariant 3-path



$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$$

because $u \in \mathcal{A}_3$ and

$$\begin{aligned}\partial u = & (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) \\ & - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2\end{aligned}$$

1.4 Homology groups

Alongside the chain complex

$$0 \stackrel{\partial}{\leftarrow} \Omega_{-1} \stackrel{\partial}{\leftarrow} \Omega_0 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \dots \tag{1.2}$$

consider also a *truncated* chain complex

$$0 \stackrel{\partial}{\leftarrow} \Omega_0 \stackrel{\partial}{\leftarrow} \dots \stackrel{\partial}{\leftarrow} \Omega_{p-1} \stackrel{\partial}{\leftarrow} \Omega_p \stackrel{\partial}{\leftarrow} \dots \tag{1.3}$$

The homology groups of (1.3) are called the *path homology groups* of the digraph G and denoted by H_p, that is,

$$H_p = \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

The homology groups of (1.2) are called the *reduced* path homology groups of G and are denoted by $\widetilde{H}_p$. We have

$$\widetilde{H}_p = H_p \text{ for } p \geq 1 \text{ and } \ \widetilde{H}_0 = H_0/\mathbb{K}.$$

Define the Betti numbers $\beta_p = \dim H_p$ and the reduced Betti numbers $\widetilde{\beta}_p = \dim \widetilde{H}_p$ so that

$$\widetilde{\beta}_p = \beta_p \text{ for } p \geq 1 \text{ and } \ \widetilde{\beta}_0 = \beta_0 - 1.$$

It is known that β_0 is equal to the number of connected components of G. In particular, if G is connected then $\widetilde{\beta}_0 = 0$.

If $G = X \sqcup Y$ - a disjoin union of two digraphs X, Y then

$$\beta_r(X \sqcup Y) = \beta_r(X) + \beta_r(Y)$$

and

$$\widetilde{\beta}_r(X \sqcup Y) = \widetilde{\beta}_r(X) + \widetilde{\beta}_r(Y) + \mathbf{1}_{\{r=0\}}.$$

1.5 Examples of spaces Ω_p and H_p

In what follows, for a vector space S over $\mathbb{K}$ we write $|S| = \dim_{\mathbb{K}} S$.

A *linear* digraph of n vertices:
$|\Omega_0| = n, \quad |\Omega_1| = n - 1,$
$\Omega_p = \{0\}$ for $p \geq 2$,
$\widetilde{H}_p = \{0\}$ for all $p \geq 0$.

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet} \longrightarrow \dots \longleftarrow \underset{n-2}{\bullet} \longrightarrow \underset{n-1}{\bullet}$$

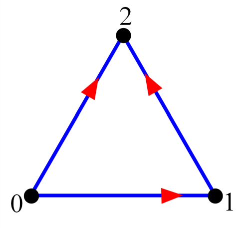
A triangle as a digraph:

$\Omega_1 = \langle e_{01}, e_{02}, e_{12} \rangle, \quad \Omega_2 = \langle e_{012} \rangle, \quad \Omega_p = \{0\}$ for $p \geq 3$

$$\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12} \rangle$$

but $$e_{01} - e_{02} + e_{12} = \partial e_{012}$$

so that $H_1 = \{0\}$. We have $\widetilde{H}_p = \{0\}$ for all $p \geq 0$.



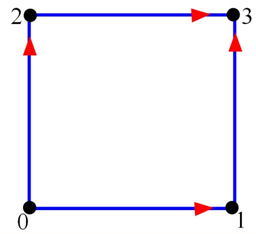
A square as a digraph:

$\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23} \rangle, \quad \Omega_2 = \langle e_{013} - e_{023} \rangle, \quad \Omega_p = \{0\}$ for $p \geq 3$

$$\ker \partial|_{\Omega_1} = \langle e_{01} + e_{13} - e_{02} - e_{23} \rangle$$

but $$e_{01} + e_{13} - e_{02} - e_{23} = \partial \left(e_{013} - e_{023} \right)$$

so that $H_1 = \{0\}$. We have $\widetilde{H}_p = \{0\}$ for all $p \geq 0$.

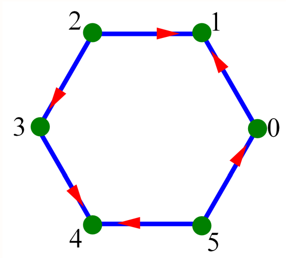


A hexagon: $|\Omega_0| = |\Omega_1| = 6, \ \Omega_p = \{0\}$ for all $p \geq 2$.

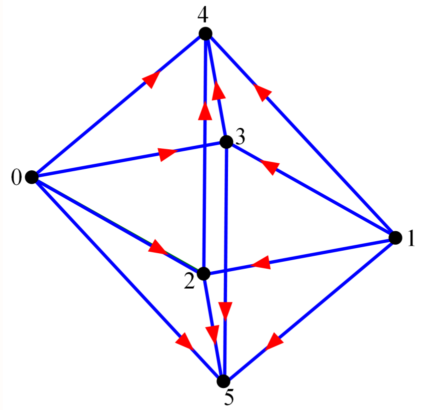
$H_1 = \langle e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50} \rangle, \quad \widetilde{H}_p = \{0\}$ for $p \neq 1$.

The same is true for any cyclic digraph (directed polygon) that is neither triangle nor square:

$$|H_1| = 1 \text{ and } \widetilde{H}_p = \{0\} \text{ for all } p \neq 1.$$

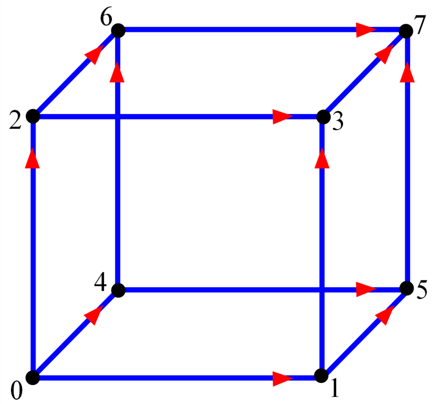


Octahedron: $|\Omega_0| = 6, \quad |\Omega_1| = 12$
Space Ω_2 is spanned by 8 triangles:
$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle,$
$|\Omega_2| = 8,\ \Omega_p = \{0\}$ for all $p \geq 3$
$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135} \rangle$
$|H_2| = 1, \quad \widetilde{H}_p = \{0\}$ for all $p \neq 2$.



A 3-cube with $|\Omega_0| = 8, \ |\Omega_1| = 12.$

$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},$$
$$e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467} \rangle$$
$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \rangle$
$\Omega_p = \{0\}$ for all $p \geq 4$
$\widetilde{H}_p = \{0\}$ for all $p \geq 0$.

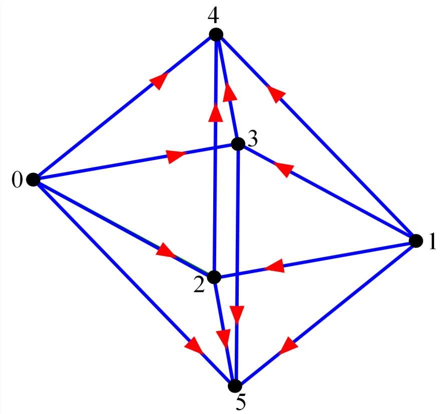


2 A join of two digraphs and the Künneth formula

Given two digraphs X, Y, define their *join* $X * Y$ as follows: take first a disjoint union $X \sqcup Y$ and add arrows from any vertex of X to any vertex of Y.

For example,

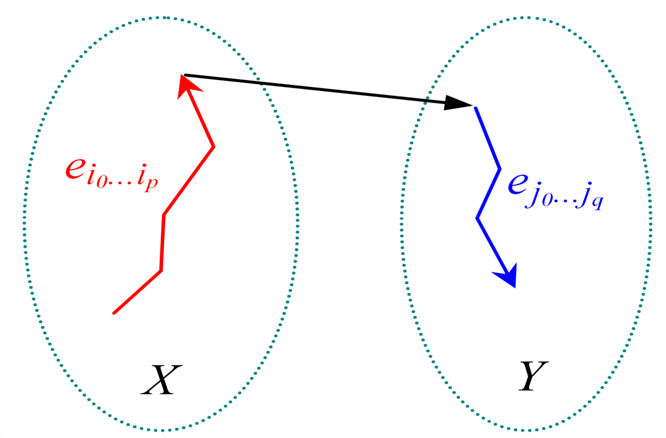
$$\{0,1\} * \{2,3\} = \begin{array}{ccc} 3 & \leftarrow & 1 \\ \uparrow & & \downarrow \\ 0 & \rightarrow & 2 \end{array} \quad \text{and} \quad \begin{array}{ccc} 3 & \leftarrow & 1 \\ \uparrow & & \downarrow \\ 0 & \rightarrow & 2 \end{array} * \{4,5\} =$$



Define the *join* uv of p-path u on X and q-path v on Y as a $(p+q+1)$-path on $X * Y$ as follows: first define it for elementary paths by

$$e_{i_0 \ldots i_p} e_{j_0 \ldots j_q} = e_{i_0 \ldots i_p j_0 \ldots j_q}$$

and then extend this definition by linearity to all p-paths u on X and q-paths v on Y.



If u and v are allowed on X resp. Y then uv is allowed on $Z = X * Y$.

Lemma 2.1 *The join of paths satisfies the product rule*

$$\partial (uv) = (\partial u) v + (-1)^{p+1} u\partial v.$$

If $u \in \Omega_p (X)$ and $v \in \Omega_q (Y)$ then ∂u and ∂v are allowed, which implies that $\partial (uv)$ is also allowed, that is, $uv \in \Omega_{p+q+1} (Z)$. The product rule implies also that the join uv is well defined for homology classes $u \in \widetilde{H}_p (X)$ and $v \in \widetilde{H}_q (Y)$ so that $uv \in \widetilde{H}_{p+q+1} (Z)$.

Theorem 2.2 (Künneth formula) *We have the following isomorphism: for any $r \geq -1$,*

$$\Omega_r(X * Y) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)) \tag{2.1}$$

that is given by the map $u \otimes v \mapsto uv$ with $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$. Consequently, for any $r \geq 0$,

$$\widetilde{H}_r(X * Y) \cong \bigoplus_{\{p,q \geq 0: p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y) \tag{2.2}$$

$$\widetilde{\beta}_r(X * Y) \cong \sum_{\{p,q \geq 0: p+q=r-1\}} \widetilde{\beta}_p(X) \widetilde{\beta}_q(Y). \tag{2.3}$$

The identity (2.1) means that any paths in $\Omega_r(Z)$ can be obtained as linear combination of joins uv where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with $p + q + 1 = r$, and (2.2) means the same for homology classes.

Note that the operation $*$ of digraphs is associative. For a sequence $X_1, ..., X_l$ of l digraphs we obtain by induction from (2.1), (2.2) and (2.3) that

$$\Omega_r(X_1 * X_2 * ... * X_l) \cong \bigoplus_{\{p_i \geq -1:\ p_1+p_2+...+p_l=r-l+1\}} \Omega_{p_1}(X_1) \otimes ... \otimes \Omega_{p_l}(X_l) \tag{2.4}$$

$$\widetilde{H}_r\left(X_1 * X_2 * \ldots * X_l\right) \cong \bigoplus_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l=r-l+1\}} \widetilde{H}_{p_1}\left(X_1\right) \otimes \ldots \otimes \widetilde{H}_{p_l}\left(X_l\right) \tag{2.5}$$

$$\widetilde{\beta}_r\left(X_1 * X_2 * \ldots * X_l\right) = \sum_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l=r-l+1\}} \widetilde{\beta}_{p_1}\left(X_1\right) \ldots \widetilde{\beta}_{p_l}\left(X_l\right). \tag{2.6}$$

Example. Consider an octahedron $Z = \underbrace{\{0,1\}}_{X_1} * \underbrace{\{2,3\}}_{X_2} * \underbrace{\{4,5\}}_{X_3}$. (see p. 13). Then

$$\begin{aligned}
\Omega_2\left(Z\right) &= \bigoplus_{\{p_i \geq -1:\ p_1+p_2+p_3=2-3+1\}} \Omega_{p_1}\left(X_1\right) \otimes \Omega_{p_2}\left(X_2\right) \otimes \Omega_{p_3}\left(X_3\right) \\
&= \Omega_0\left(X_1\right) \otimes \Omega_0\left(X_2\right) \otimes \Omega_0\left(X_3\right) = \langle e_0, e_1\rangle \otimes \langle e_2, e_3\rangle \otimes \langle e_4, e_5\rangle \\
&= \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135}\rangle
\end{aligned}$$

and

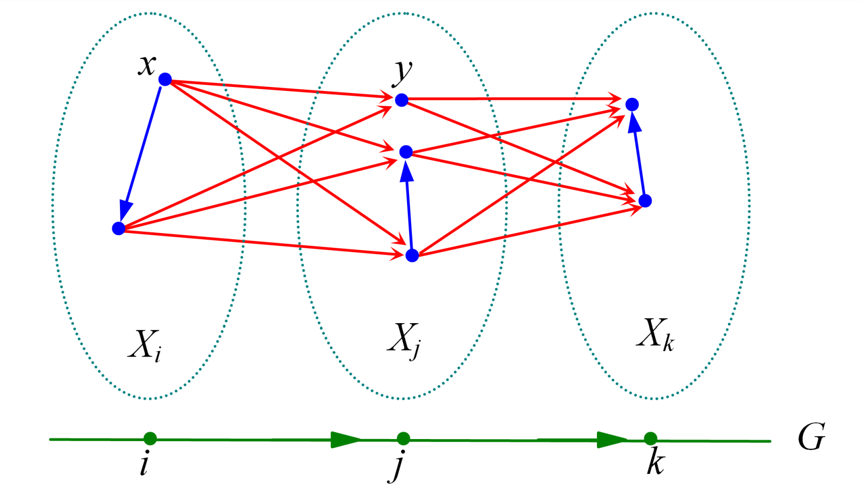
$$\begin{aligned}
H_2\left(Z\right) = \widetilde{H}_2\left(Z\right) &= \bigoplus_{\{p_i \geq 0:\ p_1+p_2+p_3=2-3+1\}} \widetilde{H}_{p_1}\left(X_1\right) \otimes \widetilde{H}_{p_2}\left(X_2\right) \otimes \widetilde{H}_{p_3}\left(X_3\right) \\
&= \widetilde{H}_0\left(X_1\right) \otimes \widetilde{H}_0\left(X_2\right) \otimes \widetilde{H}_0\left(X_3\right) = \langle e_0 - e_1\rangle \otimes \langle e_2 - e_3\rangle \otimes \langle e_4 - e_5\rangle \\
&= \langle e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135}\rangle.
\end{aligned}$$

(see p. 12).

3 A generalized join of digraphs

Given a digraph G of l vertices $\{1, 2, ..., l\}$ and a sequence $X = \{X_1, ..., X_l\}$ of l digraphs, define their *generalized join* $(X_1...X_l)_G = X_G$ as follows: X_G is obtained from the disjoint union $\bigsqcup_i X_i$ of digraphs X_i by keeping all the arrows in each X_i and by adding arrows $x \to y$ whenever $x \in X_i$, $y \in X_j$ and $i \to j$ in G.

Digraph X_G is also referred to as a *G-join* of $X_1, ..., X_l$, and G is called the *base* of X_G.



The main problem to be discussed here is

how to compute the homology groups and Betti numbers of X_G.

Denote by K_l a complete digraph with vertices $\{1, ..., l\}$ and arrows

$$i \to j \Leftrightarrow i < j$$

that is, K_l is an $(l-1)$-simplex. For example, $K_2 = \{1 \to 2\}$ and $K_3 = \begin{smallmatrix} & 2 & \\ \nearrow & & \searrow \\ 1 & \to & 3 \end{smallmatrix}$ is a triangle.

The digraph X_{K_l} is called a *complete* join of $X_1, ..., X_l$. It is easy to see that

$$X_{K_l} = X_1 * X_2 * ... * X_l$$

It follows from (2.6) that, for any $r \geq 0$,

$$\widetilde{\beta}_r(X_{K_l}) = \sum_{\{p_i \geq 0:\ p_1+p_2+...+p_l = r-l+1\}} \widetilde{\beta}_{p_1}(X_1) ... \widetilde{\beta}_{p_l}(X_l). \tag{3.1}$$

4 Homology of a monotone linear join

Denote by I_l a *monotone linear digraph* with the vertices $\{1, ..., l\}$ and arrows $i \to i+1$:

$$I_l = \{1 \to 2 \to ... \to l\}. \tag{4.1}$$

If $G = I_l$ then we use the following simplified notation:

$$(X_1 X_2 ... X_l)_{I_l} = X_1 X_2 ... X_l$$

and refer to this digraph as a *monotone linear join* of $X_1, ..., X_l$.

Clearly, $X_1 X_2 ... X_n$ can be constructed as follows: take first a disjoint union $\bigsqcup_{i=1}^{l} X_i$ and then add arrows from any vertex of X_i to any vertex of X_{i+1} (see p. 17).

In the case $l = 2$ we obviously have $X_1 X_2 = X_1 * X_2$ but in general $X_1 X_2 ... X_l$ is a subgraph of $X_1 * X_2 * ... * X_l$. For example, we have

$$\{0\}\,\{1,2\}\,\{3\} = \begin{array}{ccc} 1 & \rightarrow & 3 \\ \uparrow & & \uparrow \\ 0 & \rightarrow & 2 \end{array} \quad \text{while} \quad \{0\} * \{1,2\} * \{3\} = \begin{array}{ccc} 1 & \rightarrow & 3 \\ \uparrow & \nearrow & \uparrow \\ 0 & \rightarrow & 2 \end{array} \tag{4.2}$$

Theorem 4.1 *We have*

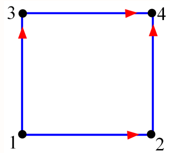
$$\widetilde{H}_r(X_1X_2...X_l) \cong \bigoplus_{\{p_i\geq 0:\ p_1+p_2+...+p_l=r-l+1\}} \widetilde{H}_{p_1}(X_1)\otimes ... \otimes \widetilde{H}_{p_l}(X_l) \tag{4.3}$$

and

$$\widetilde{\beta}_r(X_1X_2...X_l) = \sum_{\{p_i\geq 0:\ p_1+p_2+...+p_l=r-l+1\}} \widetilde{\beta}_{p_1}(X_1)...\widetilde{\beta}_{p_l}(X_l). \tag{4.4}$$

By (2.5) and (4.3), $X_1X_2...X_l$ and $X_1 * X_2 * ... * X_l$ are homologically equivalent.

Example. Let the base G be a square:



We have $G = \{1\}\{2,3\}\{4\}$ which implies that

$X_G = X_1(X_2 \sqcup X_3)X_4$. Hence, by Theorem 4.1,

$$\begin{aligned}
\widetilde{\beta}_r(X_G) &= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\widetilde{\beta}_{p_2}(X_2\sqcup X_3)\widetilde{\beta}_{p_3}(X_4) \\
&= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\left(\widetilde{\beta}_{p_2}(X_2)+\widetilde{\beta}_{p_2}(X_3)+\mathbf{1}_{\{p_2=0\}}\right)\widetilde{\beta}_{p_3}(X_4) \\
&= \widetilde{\beta}_r(X_1X_2X_4)+\widetilde{\beta}_r(X_1X_3X_4)+\widetilde{\beta}_{r-1}(X_1X_4).
\end{aligned} \tag{4.5}$$

For a general base G, if $i_1 ... i_k$ is an arbitrary sequence of vertices in G then denote

$$X_{i_1 ... i_k} = X_{i_1} X_{i_2} ... X_{i_k}.$$

Note that by (4.4)

$$\widetilde{\beta}_r \left(X_{i_1 ... i_k} \right) = \sum_{\substack{p_1 + ... + p_k = r - (k-1) \\ p_1, ..., p_k \geq 0}} \widetilde{\beta}_{p_1} \left(X_{i_1} \right) ... \widetilde{\beta}_{p_k} \left(X_{i_k} \right),$$

and we consider the numbers $\widetilde{\beta}_r \left(X_{i_1 ... i_k} \right)$ as known.

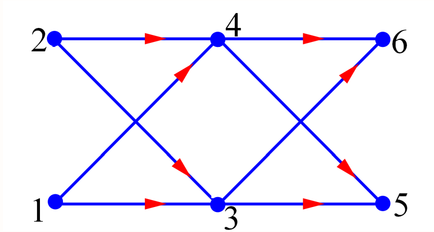
Using this notation, we can rewrite (4.5) as follows: if G is a square then

$$\widetilde{\beta}_r \left(X_G \right) = \widetilde{\beta}_r \left(X_{124} \right) + \widetilde{\beta}_r \left(X_{134} \right) + \widetilde{\beta}_{r-1} \left(X_{14} \right). \tag{4.6}$$

As we will see below, for a general base G, $\widetilde{\beta}_r \left(X_G \right)$ can always be represented as a sum of Betti numbers $\widetilde{\beta}_{r-s} \left(X_u \right)$ where $s \geq 0$ and u runs over certain paths in G.

The main problem is to determine all such paths u and their shifts s.

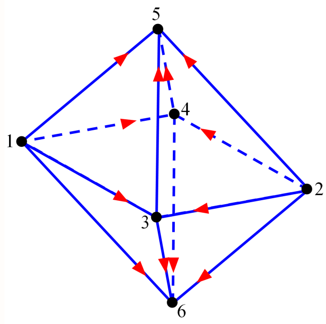
Example. Consider the following digraph G



Clearly, we have $G = \{1,2\}\,\{3,4\}\,\{5,6\}$
which implies $\quad X_G = (X_1 \sqcup X_2)\,(X_3 \sqcup X_4)\,(X_5 \sqcup X_6)$
By Theorem 4.1 we obtain

$$\begin{aligned}
\widetilde{\beta}_r(X_G) &= \sum_{\{p_i \ge 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1 \sqcup X_2)\widetilde{\beta}_{p_2}(X_3 \sqcup X_4)\widetilde{\beta}_{p_3}(X_5 \sqcup X_6) \\
&= \sum_{\{p_i \ge 0:\ p_1+p_2+p_3=r-2\}} \Big(\widetilde{\beta}_{p_1}(X_1)+\widetilde{\beta}_{p_1}(X_2)+\mathbf{1}_{\{p_1=0\}}\Big)\Big(\widetilde{\beta}_{p_2}(X_3)+\widetilde{\beta}_{p_2}(X_4)+\mathbf{1}_{\{p_2=0\}}\Big)\Big(\widetilde{\beta}_{p_3}(X_5)+\widetilde{\beta}_{p_3}(X_6)+\mathbf{1}_{\{p_3=0\}}\Big) \\
&= \widetilde{\beta}_r(X_{135})+\widetilde{\beta}_r(X_{145})+\widetilde{\beta}_r(X_{136})+\widetilde{\beta}_r(X_{146})+\widetilde{\beta}_r(X_{235})+\widetilde{\beta}_r(X_{236})+\widetilde{\beta}_r(X_{245})+\widetilde{\beta}_r(X_{246}) \\
&\quad +\widetilde{\beta}_{r-1}(X_{13})+\widetilde{\beta}_{r-1}(X_{14})+\widetilde{\beta}_{r-1}(X_{23})+\widetilde{\beta}_{r-1}(X_{24}) \\
&\quad +\widetilde{\beta}_{r-1}(X_{35})+\widetilde{\beta}_{r-1}(X_{36})+\widetilde{\beta}_{r-1}(X_{45})+\widetilde{\beta}_{r-1}(X_{46}) \\
&\quad +\widetilde{\beta}_{r-1}(X_{15})+\widetilde{\beta}_{r-1}(X_{16})+\widetilde{\beta}_{r-1}(X_{25})+\widetilde{\beta}_{r-1}(X_{26}) \\
&\quad +\widetilde{\beta}_{r-2}(X_1)+\widetilde{\beta}_{r-2}(X_2)+\widetilde{\beta}_{r-2}(X_3)+\widetilde{\beta}_{r-2}(X_4)+\widetilde{\beta}_{r-2}(X_5)+\widetilde{\beta}_{r-2}(X_6)+\mathbf{1}_{\{r=2\}}
\end{aligned}$$

Example. Let G be an octahedron:



We have $G = \{1,2\} * \{3,4\} * \{5,6\}$ whence

$$X_G = (X_1 \sqcup X_2) * (X_3 \sqcup X_4) * (X_5 \sqcup X_6)$$

By (3.1) we obtain

$$\widetilde{\beta}_r(X_G) = \sum_{\{p_i \geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1 \sqcup X_2)\widetilde{\beta}_{p_2}(X_3 \sqcup X_4)\widetilde{\beta}_{p_3}(X_5 \sqcup X_6)$$

$$= \sum_{\{p_i \geq 0:\ p_1+p_2+p_3=r-2\}} (\widetilde{\beta}_{p_1}(X_1)+\widetilde{\beta}_{p_1}(X_2)+\mathbf{1}_{\{p_1=0\}})(\widetilde{\beta}_{p_2}(X_3)+\widetilde{\beta}_{p_2}(X_4)+\mathbf{1}_{\{p_2=0\}})(\widetilde{\beta}_{p_3}(X_5)+\widetilde{\beta}_{p_3}(X_6)+\mathbf{1}_{\{p_3=0\}})$$

$$= \widetilde{\beta}_r(X_{135})+\widetilde{\beta}_r(X_{145})+\widetilde{\beta}_r(X_{235})+\widetilde{\beta}_r(X_{245})+\widetilde{\beta}_r(X_{136})+\widetilde{\beta}_r(X_{146})+\widetilde{\beta}_r(X_{236})+\widetilde{\beta}_r(X_{246})$$

$$+\widetilde{\beta}_{r-1}(X_{13})+\widetilde{\beta}_{r-1}(X_{23})+\widetilde{\beta}_{r-1}(X_{14})+\widetilde{\beta}_{r-1}(X_{24})+\widetilde{\beta}_{r-1}(X_{15})+\widetilde{\beta}_{r-1}(X_{25})$$

$$+\widetilde{\beta}_{r-1}(X_{35})+\widetilde{\beta}_{r-1}(X_{45})+\widetilde{\beta}_{r-1}(X_{16})+\widetilde{\beta}_{r-1}(X_{26})+\widetilde{\beta}_{r-1}(X_{36})+\widetilde{\beta}_{r-1}(X_{46})$$

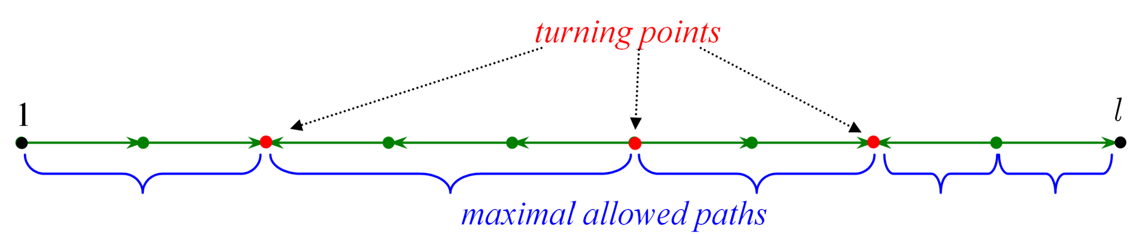
$$+\widetilde{\beta}_{r-2}(X_1)+\widetilde{\beta}_{r-2}(X_2)+\widetilde{\beta}_{r-2}(X_3)+\widetilde{\beta}_{r-2}(X_4)+\widetilde{\beta}_{r-2}(X_5)+\widetilde{\beta}_{r-2}(X_6)+\mathbf{1}_{\{r=2\}}.$$

5 Homology of linear and cyclic joins

Let now G be a *linear digraph* but not necessarily monotone. That is, the vertex set of G is $\{1, ..., l\}$ and, for any pair $(i, i+1)$ of consecutive numbers there is exactly one arrow: either $i \to i+1$ or $i \leftarrow i+1$.

Definition. We say that a vertex v of G is a *turning point* if v has either two incoming arrows or two outcoming arrows. Denote by $\mathcal{T}$ the set of all turning points.

An allowed path in G is called *maximal* if it is not a proper subset (as a set of vertices) of another allowed path. Denote by $\mathcal{A}_{\max}$ the family of all maximal allowed paths in G.



Clearly, the end vertices of a maximal path are either turning points or the vertices $1, l$.

Theorem 5.1 *If G is an arbitrary linear digraph then*

$$\widetilde{\beta}_r\left(X_G\right)=\sum_{u\in\mathcal{A}_{\max}}\widetilde{\beta}_r(X_u)+\sum_{v\in\mathcal{T}}\widetilde{\beta}_{r-1}\left(X_v\right).$$

In other words, $\widetilde{\beta}_r\left(X_G\right)$ is the sum of all $\widetilde{\beta}_r$ of the linear joins of X_i along all maximal allowed paths in G plus the sum of $\widetilde{\beta}_{r-1}$ of all X_v sitting at the turning points v.

Example. Consider a linear base

$$G=\{1\rightarrow 2\leftarrow 3\leftarrow 4\rightarrow 5\}. \tag{5.1}$$

Then $\mathcal{T}=\{2,\ 4\}$,while maximal paths of L are

$$\mathcal{A}_{\max}=\{1\rightarrow 2,\quad 4\rightarrow 3\rightarrow 2,\quad 4\rightarrow 5\}.$$

Hence, by Theorem 5.1,

$$\widetilde{\beta}_r\left(X_G\right)=\widetilde{\beta}_r\left(X_{12}\right)+\widetilde{\beta}_r\left(X_{432}\right)+\widetilde{\beta}_r\left(X_{45}\right)+\widetilde{\beta}_{r-1}\left(X_2\right)+\widetilde{\beta}_{r-1}\left(X_4\right).$$

Example. Consider the following base G:

$$G = \begin{array}{ccccccccc} & & 2 & & & & 5 & & \\ & \nearrow & & \nwarrow & & \swarrow & & \searrow & \\ 1 & & & & 3 & & & & 6 \\ & \searrow & & \swarrow & & \nwarrow & & \nearrow & \\ & & 4 & & & & 7 & & \end{array}$$

$$\Gamma = \quad \mathrm{i} \longrightarrow \mathrm{ii} \longleftarrow \mathrm{iii} \longleftarrow \mathrm{iv} \longrightarrow \mathrm{v}$$

It is easy to see that G itself is the following linear join:

$$G = (\{1\}\{2,4\}\{3\}\{5,7\}\{6\})_\Gamma$$

where the digraph Γ is the same as in (5.1). The turning points of Γ are $\mathcal{T} = \{\mathrm{ii},\ \mathrm{iv}\}$, and maximal paths of Γ are

$$\mathcal{A}_{\max} = \{\mathrm{i} \to \mathrm{ii},\ \ \mathrm{iv} \to \mathrm{iii} \to \mathrm{ii},\ \ \mathrm{iv} \to \mathrm{v}\}.$$

For a Γ-join Y_Γ we have as above

$$\widetilde{\beta}_r(Y_\Gamma) = \widetilde{\beta}_r(Y_{\mathrm{i,ii}}) + \widetilde{\beta}_r(Y_{\mathrm{iv,iii,ii}}) + \widetilde{\beta}_r(Y_{\mathrm{iv,v}}) + \widetilde{\beta}_{r-1}(Y_{\mathrm{ii}}) + \widetilde{\beta}_{r-1}(Y_{\mathrm{iv}}).$$

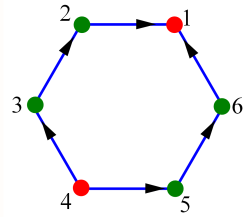
Setting $Y_{\mathrm{i}} = X_1$, $Y_{\mathrm{ii}} = X_2 \sqcup X_3$, $Y_{\mathrm{iii}} = X_3$, $Y_{\mathrm{iv}} = X_5 \sqcup X_7$ and $Y_{\mathrm{v}} = X_6$ we obtain

$$
\begin{aligned}
\widetilde{\beta}_r\left(X_G\right) &= \widetilde{\beta}_r\left(\left(X_1\left(X_2 \sqcup X_3\right) X_3\left(X_5 \sqcup X_7\right) X_6\right)_\Gamma\right) \\
&= \widetilde{\beta}_r(X_1(X_2 \sqcup X_4)) + \widetilde{\beta}_r((X_5 \sqcup X_7)X_3(X_2 \sqcup X_4)) + \widetilde{\beta}_r((X_5 \sqcup X_7)X_6) \\
&\qquad\qquad + \widetilde{\beta}_{r-1}(X_2 \sqcup X_4) + \widetilde{\beta}_{r-1}(X_5 \sqcup X_7) \\
&= \widetilde{\beta}_r\left(X_{12}\right) + \widetilde{\beta}_r\left(X_{14}\right) + \widetilde{\beta}_{r-1}\left(X_1\right) \\
&+ \widetilde{\beta}_r(X_{532}) + \widetilde{\beta}_r(X_{534}) + \widetilde{\beta}_r(X_{732}) + \widetilde{\beta}_r(X_{734}) \\
&\qquad\qquad + \widetilde{\beta}_{r-1}(X_{32}) + \widetilde{\beta}_{r-1}(X_{34}) + \widetilde{\beta}_{r-1}(X_{53}) + \widetilde{\beta}_{r-1}(X_{73}) + \widetilde{\beta}_{r-2}(X_3) \\
&+ \widetilde{\beta}_r\left(X_{56}\right) + \widetilde{\beta}_r\left(X_{76}\right) + \widetilde{\beta}_{r-1}\left(X_6\right) \\
&+ \widetilde{\beta}_{r-1}\left(X_2\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}} + \widetilde{\beta}_{r-1}\left(X_5\right) + \widetilde{\beta}_{r-1}\left(X_7\right) + \mathbf{1}_{\{r=1\}}.
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\beta}_r(X_G) &= \widetilde{\beta}_r(X_{534}) + \widetilde{\beta}_r(X_{532}) + \widetilde{\beta}_r(X_{734}) + \widetilde{\beta}_r(X_{732}) \\
&+ \widetilde{\beta}_r(X_{12}) + \widetilde{\beta}_r(X_{14}) + \widetilde{\beta}_r(X_{56}) + \widetilde{\beta}_r(X_{76}) \\
&+ \widetilde{\beta}_{r-1}(X_{73}) + \widetilde{\beta}_{r-1}(X_{53}) + \widetilde{\beta}_{r-1}(X_{32}) + \widetilde{\beta}_{r-1}(X_{34}) \\
&+ \widetilde{\beta}_{r-1}(X_1) + \widetilde{\beta}_{r-1}(X_2) + \widetilde{\beta}_{r-1}(X_4) + \widetilde{\beta}_{r-1}(X_5) + \widetilde{\beta}_{r-1}(X_6) + \widetilde{\beta}_{r-1}(X_7) \\
&+ \widetilde{\beta}_{r-2}(X_3) + \mathbf{2}_{\{r=1\}}.
\end{aligned}
$$

A digraph G is called *cyclic* if it is connected and each vertex has the undirected degree 2. Let G be a cyclic digraph with the set of vertices $V = \{1, 2, ..., l\}$. We assume that the vertices are ordered so that every vertex $i \in V$ is connected by arrows to $i-1$ and $i+1$ (where l is identified with 0). In the same way as above we define the set $\mathcal{A}_{\max}$ and $\mathcal{T}$.

For example, consider the following hexagon:
Here $\mathcal{T} = \{1, 4\}$ and
$\mathcal{A}_{\max} = \{4 \to 3 \to 2 \to 1,\ 4 \to 5 \to 6 \to 1\}$



Theorem 5.2 *Let G be a cyclic digraph that is neither triangle nor square nor double arrow. Then*

$$\widetilde{\beta}_r(X_G) = \sum_{u \in \mathcal{A}_{\max}} \widetilde{\beta}_r(X_u) + \sum_{v \in \mathcal{T}} \widetilde{\beta}_{r-1}(X_v) + \widetilde{\beta}_r(G). \tag{5.2}$$

Note that in this case $\widetilde{\beta}_r(G) = \mathbf{1}_{\{r=1\}}$. If G is a triangle or square or double arrow then (5.2) is wrong, which is shown in Examples below.

Example. If G is the above hexagon then we obtain

$$\widetilde{\beta}_r\left(X_G\right) = \widetilde{\beta}_r\left(X_{4321}\right) + \widetilde{\beta}_r\left(X_{4561}\right) + \widetilde{\beta}_{r-1}\left(X_1\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}}.$$

Example. Consider the following 4-cycylic base:

$$G = \begin{array}{ccc} 2 & \longrightarrow & 3 \\ \uparrow & & \downarrow \\ 1 & \longrightarrow & 4 \end{array}$$

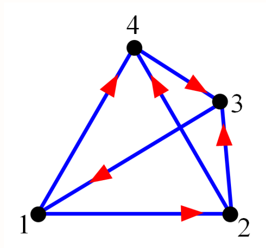
Since $\mathcal{T} = \{1,4\}$ and $\mathcal{A}_{\max} = \{1 \to 2 \to 3 \to 4, 1 \to 4\}$, we obtain

$$\widetilde{\beta}_r(X_G) = \widetilde{\beta}_r(X_{1234}) + \widetilde{\beta}_r(X_{14}) + \widetilde{\beta}_{r-1}\left(X_1\right) + \widetilde{\beta}_{r-1}\left(X_4\right) + \mathbf{1}_{\{r=1\}}. \tag{5.3}$$

Example. Consider the following 3-cyclic base: $G = \begin{tikzcd} & \overset{2}{\bullet} \arrow[dr] & \\ \overset{1}{\bullet} \arrow[ur] & & \overset{3}{\bullet} \arrow[ll] \end{tikzcd}$.

Then $\mathcal{A}_{\max}$ and $\mathcal{T}$ are empty, and we obtain $\widetilde{\beta}_r\left(X_G\right) = \mathbf{1}_{\{r=1\}} = \widetilde{\beta}_r(G)$.

Example. Consider the following tetrahedron as a base G:



We have $\quad G = C * \{4\} \quad$ where

$$C = \{1 \to 2 \to 3 \to 1\}$$

It follows that

$$X_G = X_C * X_4$$

and

$$\widetilde{\beta}_r (X_G) = \sum_{p+q=r-1} \widetilde{\beta}_p (X_C) \widetilde{\beta}_q (X_4) = \sum_{p+q=r-1} \mathbf{1}_{\{p=1\}} \widetilde{\beta}_q (X_4) = \widetilde{\beta}_{r-2} (X_4) \, .$$

Hence, $\widetilde{\beta}_r (X_G) = \widetilde{\beta}_{r-2} (X_4)$.

Example. Let G be a triangle: $G = \begin{array}{ccc} & \overset{2}{\bullet} & \\ \nearrow & & \searrow \\ {}^{1}\bullet & \longrightarrow & \bullet^{3} \end{array}$. Then $X_G = X_1 * X_2 * X_3$ and we know that

$$\widetilde{\beta}_r (X_G) = \widetilde{\beta}_r (X_{123}) \, .$$

However, the right hand side of (5.2) is in this case

$$\widetilde{\beta}_r (X_{123}) + \widetilde{\beta}_{r-1} (X_1) + \widetilde{\beta}_{r-1} (X_3) \neq \widetilde{\beta}_r (X_G) \, .$$

Example. Let G be a square:

$$G=\begin{array}{ccc}3&\rightarrow&4\\\uparrow&&\uparrow\\1&\rightarrow&2\end{array}$$

Then we that by (4.6)

$$\widetilde{\beta}_r\left(X_G\right)=\widetilde{\beta}_r\left(X_{124}\right)+\widetilde{\beta}_r\left(X_{134}\right)+\widetilde{\beta}_{r-1}\left(X_{14}\right),$$

while the right hand side of (5.2) is in this case

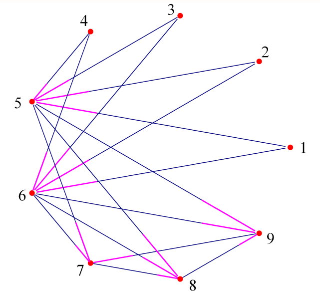
$$\widetilde{\beta}_r\left(X_{124}\right)+\widetilde{\beta}_r\left(X_{134}\right)+\widetilde{\beta}_{r-1}\left(X_1\right)+\widetilde{\beta}_{r-1}\left(X_4\right).$$

Example. Let G be a double arrow: $G=\{1\leftrightarrows 2\}$. Then

$$X_G=X_1*X_2*X_1$$

whence $\widetilde{\beta}_r\left(X_G\right)=\widetilde{\beta}_r\left(X_{121}\right)$. However, in this case $\mathcal{A}_{\max}$ and $\mathcal{T}$ are empty, so that the right hand side of (5.2) is $\widetilde{\beta}_r\left(G\right)=0$.

Example. Let G be as here:



We have
$G = \{1,2,3,4\}\{5,6\}\{7 \to 8 \to 9 \to 7\}$

so that
$X_G = (X_1 \sqcup X_2 \sqcup X_3 \sqcup X_4)(X_5 \sqcup X_6) X_{\{7 \to 8 \to 9 \to 7\}}$

It follows that

$$\widetilde{\beta}_r(X_G) = \sum_{p+q+s=r-2} \left(\widetilde{\beta}_p(X_1) + \widetilde{\beta}_p(X_2) + \widetilde{\beta}_p(X_3) + \widetilde{\beta}_p(X_4) + \mathbf{3}_{\{p=0\}}\right) \times \left(\widetilde{\beta}_q(X_5) + \widetilde{\beta}_q(X_6) + \mathbf{1}_{\{q=0\}}\right) \mathbf{1}_{\{s=1\}}$$

which yields after computation

$$\begin{aligned}\widetilde{\beta}_r(X_G) = {} & \widetilde{\beta}_{r-2}(X_{15}) + \widetilde{\beta}_{r-2}(X_{16}) + \widetilde{\beta}_{r-2}(X_{25}) + \widetilde{\beta}_{r-2}(X_{26}) \\ & + \widetilde{\beta}_{r-2}(X_{35}) + \widetilde{\beta}_{r-2}(X_{36}) + \widetilde{\beta}_{r-2}(X_{45}) + \widetilde{\beta}_{r-2}(X_{46}) \\ & + \widetilde{\beta}_{r-3}(X_1) + \widetilde{\beta}_{r-3}(X_2) + \widetilde{\beta}_{r-3}(X_3) + \widetilde{\beta}_{r-3}(X_4) + 3\widetilde{\beta}_{r-3}(X_5) + 3\widetilde{\beta}_{r-3}(X_6) + \mathbf{3}_{\{r=3\}}.\end{aligned}$$

6 Homology of a generalized join

Theorem 6.1 *There exists a finite sequence of paths $\{u_k\}$ in G and a sequence $\{s_k\}$ of non-negative integers such that, for any sequence $\{X_i\}$ of digraphs and any $r \geq 0$,*

$$\widetilde{\beta}_r(X_G) = \sum_k \widetilde{\beta}_{r-s_k}(X_{u_k}) + \widetilde{\beta}_r(G). \tag{6.1}$$

Besides, the sequence $\{u_k\}$ contains all maximal allowed paths, and $u_k \in \mathcal{A}_{\max} \Leftrightarrow s_k = 0$.

Example. Let the base G be a cube.

Using description of paths u_k from the proof of Theorem 6.1, we obtain

$$\begin{aligned}\widetilde{\beta}_r(X_G) = {} & \widetilde{\beta}_r(X_{1248}) + \widetilde{\beta}_r(X_{1268}) + \widetilde{\beta}_r(X_{1348}) \\ & + \widetilde{\beta}_r(X_{1378}) + \widetilde{\beta}_r(X_{1568}) + \widetilde{\beta}_r(X_{1578}) \\ & + \widetilde{\beta}_{r-1}(X_{178}) + \widetilde{\beta}_{r-1}(X_{168}) + \widetilde{\beta}_{r-1}(X_{148}) \\ & + \widetilde{\beta}_{r-1}(X_{128}) + \widetilde{\beta}_{r-1}(X_{138}) + \widetilde{\beta}_{r-1}(X_{158}) \\ & + \widetilde{\beta}_{r-2}(X_{18})\end{aligned}$$