Lecture 18 16 May 2022

8 lectures remain: Compactification (2 lectures)

Rough isometry (3-4 lectures).

Strong isoperimetric ineq... (2 lectures)

Ch. VI Compactification.

Def BD = space of bounded Dirichlet functions.

where $\|u\|_{BD} = \sup_{x \in V} |u(x)| + (D(u))^{\frac{1}{2}}$

Here $D(u) = \frac{1}{2} \sum c_{xy} (u_x - u_y)^2$

① BD is better than D since their restriction to "boundary" is a well-defined function.

② BHD is dense in HD with respect $\|\cdot\|_D$

$$BHD := BD \cap HD \quad , \quad BD_0 = BD \cap D_0$$

Let $f, g \in \cancel{BHD}$, BD

$$(fg)(x) := f(x)\, g(x)$$

$$\Rightarrow \quad fg \in BD \quad ?$$

Check: ① fg is bounded

since $\sup|fg| \le \sup|f| \sup|g|$

② fg has finite energy,

Let $\alpha := \sup|f|$, $\beta := \sup|g|$,

$$c(x,y)\left[(fg)(x) - (fg)(y)\right] = c_{xy}\left|f(x)g(x) - f(y)g(y)\right|$$

$$= c_{xy}\left|\left(\frac{f(x)+f(y)}{2}\right)(g(x)-g(y)) + \frac{(g(x)+g(y))}{2}(f(x)-f(y))\right|$$

$$\le \alpha\, c_{xy}\,|g(x)-g(y)| + \beta\, c_{xy}\,(f_x - f_y)$$

$$\Rightarrow \quad D(fg) \leq \left(\alpha \left(D(g)\right)^{\frac{1}{2}} + \beta \left(D(f)\right)^{\frac{1}{2}} \right)^2$$

Thm BD is commutative algebra with respect to the pointwise product. The subspace BD_0 is ideal of BD,

[i.e. for any $f \in BD_0$ and $g \in BD$, we have $fg \in BD_0$]

Remark: ~~Generally~~ if $f, g \in D$, generally ~~then~~ $fg \notin D$.

<u>Def</u> BD the Dirichlet algebra of (Γ, r).

Recall: Royden decomposition

$$D = HD \oplus D_0$$

$$BD = BHD \oplus BD_0$$

<u>Thm</u> Let (Γ, r) is transient and $F \subset V$ subset of vertices (possibly empty).

Then for any $f \in BD$, $\exists!$ u, v s.t.

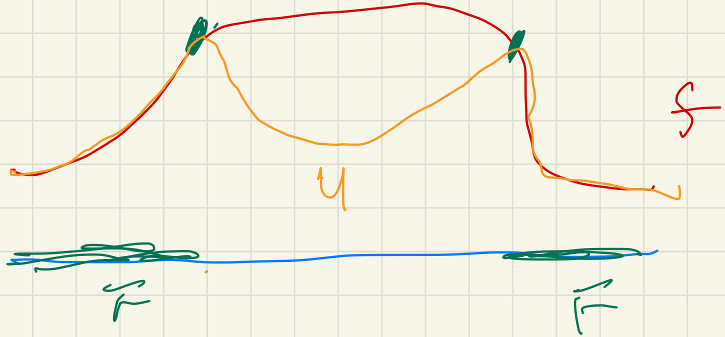
(1) $f = u + v$

(2) $u \in BD$ and harmonic over $V - F$

(3). $v = 0$ on F and v can be approximated by functions with finite support in $V - F$.

$v = f - u$

Idea:



$\mathbb{R} \sim (\Gamma, r)$

$f\big|_{\partial F}$

Proof is same as Royden's decomposition for D.

Royden's compactification

Thm Let (Γ, r) be electric network. There exists a unique (up to homeomorphism) compact Hausdorff space R s.t.

(weak* - topology
= pointwise - convergence)

(1), $V \overset{\text{injective.}}{\hookrightarrow} R$ and V is open dense set in R.

(2), every function in BD can be continuously extended to R.

(3), BD separates points in R, i.e. for any $x, y \in R$ $\exists f$ s.t. $f(x) \neq f(y)$.

Construct $\mathbb{R}$ using functional analysis.

Recall: If V is normed vector space,

then V^* dual space = space of all bounded linear functional.

$\Rightarrow$ V^* normed vector space.

$\Rightarrow$ $V^{**} = (V^*)^*$, V^{***}

There is an injection $V \hookrightarrow V^{**}$ given as follow:

Let $\underline{x} \in V$. Define $\hat{x}$ s.t. for any $f \in V^*$

$$\hat{x}(f) := f(x)$$

$\Rightarrow$ $\hat{x}$ is linear over V^*

Check: $\hat{x}(\alpha f+\beta g) = (\alpha f+\beta g)(x) = \alpha\hat{x}(f)+\beta\hat{x}(g)$.

V BD BD^*

$\Downarrow$

$f: V\to\mathbb{R}$

bounded with finite energy

Pf: $V \hookrightarrow BD^*$

For any $x\in V$, we define $\hat{x}\in BD^*$ s.t.

$$\hat{x}(f) = f(x) \quad \text{for all } f\in BD.$$

We define R to be the compact space of all non-trivial multiplicative linear functions on BD.

i.e. $\hat{x} \in R$ is multiplicative if

$$\hat{x}(fg) = \hat{x}(f) \cdot \hat{x}(g) \quad \text{for } f, g \in BD.$$

Claim: $V \overset{\text{check?}}{\subset} R \overset{\text{know}}{\subset} BD^*$

Check: Pick $x \in V$.

$$\hat{x}(fg) = (fg)(x) = f(x)\,g(x)$$
$$= \hat{x}(f)\,\hat{x}(g)$$

$\Rightarrow$ $\hat{x}$ is multiplicative a

and non-trivial since $\hat{x}(\delta_x) = \delta_x(x) = 1$

$\Rightarrow$ $\hat{x} \in R$.

$\Rightarrow$ $V \hookrightarrow R \subset BD^*$

Remark from functional analysis:

① $R = \left\{ \text{non-trivial } \cancel{\text{multiplicative}} \text{ linear on } BD. \right\}$ $\longleftrightarrow$ compact in weak* topology.

② The same construction can be applied to any normed vector space $\mathcal{U}$ of functions over V.

e.g. $\mathcal{U} = BD$ $\Rightarrow$ Royden's compactification

$\mathcal{U}$ = span of green's functions $\Rightarrow$ Martin boundary

$$V \hookrightarrow \hat{R} \subset \mathcal{U}^*$$

compact

Def R Royden's compactification of (Γ, r),

$bR := R - V$ Royden's boundary,

$\Delta = \{ x \in bR \subset \mathbb{R}^{BD^*} \mid \underset{\hat{x}(f)}{\overset{\shortparallel}{f(x)}} = 0 \quad \forall f \in BD_0 \}$, is

called harmonic boundary of (Γ, r).

$$R = V + bR$$

$$bR = \Delta + (bR - \Delta)$$

Thm $\quad \Delta = \phi \iff (\Gamma, r)$ is recurrent.

Pf: $(\Leftarrow)$ Assume (Γ, r) is recurrent.

e const. function $\in D_0 \Rightarrow D = D_0$

$\Rightarrow BD = BD_0$

let $\hat{x} \in BD^*$, s.t. $\hat{x}(f) = 0 \quad \forall f \in BD_0 = BD$

$\Rightarrow \hat{x}(f) = 0 \quad \forall f \in BD$

$\Rightarrow \hat{x} = 0$ is trivial as an element in BD^*

$\Rightarrow \hat{x} \notin R$

$\Rightarrow \quad \Delta = \phi$

Converse statement holds with some analysis

using weak* topology. $\square$

Corollary: $\Delta \neq \phi \iff (\Gamma, r)$ is transient.