$A^p(X;K_n^M)=H^p(C^*(X;K_n^M))$ $\qquad$ $C^i(X;K_n^M):=\bigoplus_{x\in X^{(i)}} K^M_{n-i}(k(x))$

$H^{p,q}(X,\mathbb{Z})$ $p\geq 2q-1$

Prop 8.1 Define $\pi: X\times\mathbb{A}^1\to X$ the projection. Then the pull-back $\pi^*: A^p(X;K_n^M)\to A^p(X\times\mathbb{A}^1;K_n^M)$ is an isomorphism.

Proof: Define $C^{i,p}(\pi)=\bigoplus_{\substack{x\in (X\times\mathbb{A}^1)^{(p)}\\ \operatorname{codim}_X\overline{\pi(x)}\geq i}} K^M_{n-p}(k(x))\subseteq C^p(X\times\mathbb{A}^1;K_n^M)$. Then we have a finite filtration $\cdots\subseteq C^{i,p}(\pi)\subseteq C^{i-1,p}(\pi)\subseteq\cdots$ of $C^p(X\times\mathbb{A}^1;K_n^M)$, where

$$C^{i,p}(\pi)/C^{i+1,p}(\pi)=\bigoplus_{u\in X^{(i)}} C^{p-i}(\mathbb{A}^1_{k(u)};K_n^M).$$

So we obtain a spectral sequence $E_1^{a,b}(\pi)=\bigoplus_{u\in X^{(a)}} A^b(\mathbb{A}^1_{k(u)};K_n^M)\Rightarrow A^{a+b}(X\times\mathbb{A}^1;K_n^M)$

We have $A^b(\mathbb{A}^1_{k(u)};K_n^M)=\begin{cases}0 & b\neq 0\\ K_n^M(k(u)) & b=0\end{cases}=A^b(k(u);K_n^M)$ by Thm 3.4.

So the spectral sequence degenerates to the complex $C^*(X;K_n^M)=E_1^{*,0}(\pi)$. Hence the statement follows. □

Cor 8.2 Suppose $\pi: V\to X$ is an $\mathbb{A}^n$-bundle. The π^* is an isomorphism.

Thm 8.3 (Gabber's representation theorem) Let F be a field and X be the localization of a smooth F-scheme and $\dim X=n$. Suppose $Y\subseteq X$ is a closed subscheme of codimension ≥ 1. Then there is a closed point $t\in\mathbb{A}^{n-1}_F$ and an étale morphism $\pi: X\to\mathbb{A}^1\times S$, where $S=\operatorname{Spec}(\mathcal{O}_{\mathbb{A}^{n-1}_F,t})$, s.t. the composite $Y\subseteq X\xrightarrow{\pi}\mathbb{A}^1\times S$ is still a closed immersion and finite over S. Moreover, one can require that $Y=\pi^{-1}(\pi(Y))$.

$$\begin{tikzcd}
Y \arrow[r, phantom, "\subseteq"] \arrow[drr, "\text{finite}"'] & X \arrow[r, "\pi"] & \mathbb{A}^1\times S \arrow[d] \\
 & & S
\end{tikzcd}$$

Proof: See [Colliot-Thélène, Hoobler, Kahn, The Bloch-Ogus-Gabber theorem, Theorem 3.1.1] when F is infinite. See [Hogadi, Kulkarni, Gabber's representation lemma for finite fields] when F is finite. □

Thm 8.4 Suppose $X=\operatorname{Spec}\mathcal{O}_{Y,y}$ where $Y\in Sm/k$. Then

$$A^p(X;K_n^M)=0$$

for every $p>0$.

Proof: We prove by induction on $\dim X$. If $\dim X=0$, the statement is trivial. So let $r\in Z^i(C^*(X;K_n^M))$, $i>0$. So $r=\alpha_1+\cdots+\alpha_r$, where $\alpha_j\in K^M_{n-i}(k(y_j))$, $y_j\in X^{(i)}$. By Thm 8.3, one may choose an étale morphism $\underbrace{f: X\to\mathbb{A}^1_S}$ s.t. $S=\operatorname{Spec}\mathcal{O}_{\mathbb{A}^{\dim X-1},t}$ and $\overline{y_j}\subseteq X\xrightarrow{f}\mathbb{A}^1_S$ are closed immersions. So in particular the map f is Nisnevich, $f^{-1}(f(y_j))=y_j$.

So we can find $r'=\alpha_1'+\cdots+\alpha_r'\in Z^i(C^*(\mathbb{A}^1_S;K_n^M))$ s.t. $f^*(r')=r$ where $\alpha_j'\in K^M_{n-i}(k(f(y_j)))=K^M_{n-i}(k(y_j))$. So it suffices to show that r' is a boundary. But by Prop 8.1, $H^i(C^*(\mathbb{A}^1_S;K_n^M))=\underbrace{H^i(C^*(S;K_n^M))}$ and the latter is zero since $\dim S<\dim X$. So we are done. □

Now for every closed embedding $i: Y\to X$ of smooth schemes, we want to define the Gysin map $i^*: A^p(X;K_n^M)\to A^p(Y;K_n^M)$.

We define the deformation space $D(X,Y)$ as follows:

$$D(X,Y)=Bl_{Y\times 0}(X\times\mathbb{A}^1)\setminus Bl_{Y\times 0}(X\times 0).$$

It is smooth with a closed embedding $j: Y\times\mathbb{A}^1\to D(X,Y)$ and has a flat morphism $\rho: D(X,Y)\to\mathbb{A}^1$, s.t. the diagram commutes

$$\begin{tikzcd}
Y\times\mathbb{A}^1 \arrow[r, "j"] \arrow[dr, "p_2"'] & D(X,Y) \arrow[d, "\rho"] \\
 & \mathbb{A}^1
\end{tikzcd}$$

Moreover, we have

(1) $\rho^{-1}(\mathbb{A}^1\setminus 0)=X\times(\mathbb{A}^1\setminus 0)$ and after restricting on j, it is the embedding $i\times id: Y\times(\mathbb{A}^1\setminus 0)\to X\times(\mathbb{A}^1\setminus 0)$.

(2) $\rho^{-1}(0)=N_{Y/X}$, where $N_{Y/X}$ is the normal bundle and the restriction on j is the zero section $s_0: Y\to N_{Y/X}$.

This process is know as "deformation to normal bundle".

Def 8.5 Let X be finite type $/k$. Let $i: Y\to X$ be a closed immersion and $j: U=X\setminus Y\to X$ be the inclusion. Define

$$\partial: C^p(U;K_n^M)\to C^{p+1-c}(Y;K^M_{n-c}) \qquad c=d_X-d_Y$$

$$\forall x\in U^{(p)}, y\in Y^{(p+1-c)},\ \partial_y^x=\begin{cases}\partial_y^x & \text{if } y\in\bar{x}\\ 0 & \text{else}\end{cases}$$

$d_X\circ d_X=0$

By Thm 3.18, the ∂ induces a map $\partial: A^p(U;K_n^M)\to A^{p+1-c}(Y;K^M_{n-c})$. Since there is a decomposition $C^p(X;K_n^M)=C^{p-c}(Y;K^M_{n-c})\oplus C^p(U;K_n^M)$, we have a long exact sequence

$$\cdots\to A^{p+c}(U;K^M_{n+c})\xrightarrow{\partial}A^{p+1}(Y;K_n^M)\xrightarrow{i_*}A^{p+1+c}(X;K^M_{n+c})\xrightarrow{j^*}A^{p+1+c}(U;K^M_{n+c})\to\cdots$$

Similarly, for any $t\in\mathcal{O}^*(X)$, the multiplication by $[t]$ induces a map

$$[t]: A^p(X;K_n^M)\to A^p(X;K^M_{n+1}).$$

Now we define $i^*: A^p(X;K_n^M)\to A^p(Y;K_n^M)$ by the composite

$$A^p(X;K_n^M)\to A^p(X\times(\mathbb{A}^1\setminus 0);K_n^M)\xrightarrow[\substack{\downarrow\\ \text{parameter}\\ \text{of } \mathbb{A}^1\setminus 0}]{[t]}A^p(X\times(\mathbb{A}^1\setminus 0);K^M_{n+1})\xrightarrow{\partial}A^p(N_{Y/X};K_n^M)\xrightarrow[\substack{\downarrow\\ \text{Cor } 8.2}]{\cong}A^p(Y;K_n^M).$$

Def 8.6 Suppose X,Y are f.t.$/k$. For every $x\in X^{(n)}$, $y\in Y^{(m)}$ and $z\in(\bar{x}\times\bar{y})^{(0)}$, define

$$\begin{aligned}\times: K_a^M(k(x))\times K_b^M(k(y)) &\to K^M_{a+b}(k(z))\\ (u,v) &\mapsto p_1^*(u)_z\cdot p_2^*(v)_z\end{aligned}$$

$$\begin{tikzcd}
X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\
X &
\end{tikzcd}$$

This induces an exterior product

$$\times: C^p(X;K_n)\times C^q(Y;K_m^M)\to C^{p+q}(X\times Y;K^M_{n+m}).$$

Prop 8.7 For any $\rho\in C^p(X;K_n^M)$, $\mu\in C^q(Y;K_m^M)$, we have

$$d_{X\times Y}(\rho\times\mu)=d_X(\rho)\times\mu+(-1)^p\rho\times d_Y(\mu).$$

Proof: Suppose $x\in X^{(a)}$, $y\in Y^{(b)}$ and z is a generic point of $\bar{x}\times\bar{y}$. For any $w\in Z^{(1)}$, it corresponds to $(w_x,w_y)\in\bar{x}\times\bar{y}$. Then a dimension argument shows that either $w_x\in\bar{x}^{(1)}$, $w_y\in\bar{y}^{(0)}$ or $w_x\in\bar{x}^{(0)}$, $w_y\in\bar{y}^{(1)}$. So let us consider the first case and suppose $\rho\in K^M_{n-p}(k(x))$, $\mu\in K^M_{m-q}(k(y))$. We have a diagram

$$\begin{tikzcd}
 & w \arrow[d] & w_i \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\
 & Z \arrow[d, phantom, "\cap"] & \tilde{Z} \arrow[l] \arrow[dd, "p_2"] \\
 & \bar{x}\times\bar{y} \arrow[dl] \arrow[d] & \\
\bar{x} & \bar{y} & \tilde{y} \arrow[l]
\end{tikzcd}$$

where $\tilde{Z}$, $\tilde{y}$ are normalizations and $\{w_i\}$ are fibers of w. For any $f\in k(y)$, $p_2^*(f)$ has zero valuation on w_i. Since if $p_2^*(f)=0$ generically on w_i, then $f=0$ in $k(y)$. By construction in Prop 3.3, the ∂_i w.r.t. w_i satisfies (π is a uniformizer)

$$\forall a,b\in K_*^M(k(Z)),\ \underline{\partial_i(a\cdot b)}=\partial_i(a)\cdot \underline{\underline{s^\pi}}(b)+(-1)^{\deg(a)}s^\pi(a)\cdot\partial_i(b)+\partial_i(a)\partial_i(b)\{-1\}$$

So $\partial_i(p_2^*(\mu))=0$ gives $\partial_i(p_1^*(\rho)\cdot p_2^*(\mu))=\partial_i(p_1^*(\rho))\cdot s^\pi(p_2^*(\mu))$.

Hence

$$\begin{aligned}
d_{X\times Y}(\rho\times\mu)_w &= \sum_i N_{k(w_i)/k(w)}(\partial_i(p_1^*(\rho)p_2^*(\mu)))\\
&= \sum_i N_{k(w_i)/k(w)}(\partial_i(p_1^*(\rho))\cdot \underbrace{s^\pi(p_2^*(\mu))}_{\in K_*^M(k(w))})\\
&= \sum_i N_{k(w_i)/k(w)}(\partial_i(p_1^*(\rho)))\cdot s^\pi(p_2^*(\mu))\\
&= (d_X(\rho)\times\mu)_w. \qquad \square
\end{aligned}$$

Cor 8.8 There is an exterior product

$$\times: A^p(X;K_n^M)\times A^q(Y;K_m^M)\to A^{p+q}(X\times Y;K^M_{n+m})$$

for f.t.$/k$ X, Y.

Now for every $f: X\to Y$, we can write it as $X\xrightarrow{\Gamma_f}X\times Y\xrightarrow{p_2}Y$. Then define $f^*: A^p(Y;K_n^M)\to A^p(X;K_n^M)$ as $\Gamma_f^*p_2^*$. This definition is functorial (See [Rost, Chow groups with coefficients, Theorem 12.1]).

On the other hand, for any $X\in Sm/k$, we obtain an intersection product by composite

$$\underset{x}{\underset{\in}{A^p(X;K_n^M)}}\times \underset{y}{\underset{\in}{A^q(X;K_m^M)}}\xrightarrow{\times}A^{p+q}(X\times X;K^M_{n+m})\xrightarrow{\delta^*}A^{p+q}(X;K^M_{n+m}).$$

This product is associative and graded commutative, i.e. $x\cdot y=(-1)^{(n-p)(m-q)}y\cdot x$. (See [Rost, 14.2, 14.3]).

Prop 8.9 The $A^p(-;K_n^M)$ is a homotopy invariant presheaf with transfers.

Proof: We have products and pull-backs. For any elementary correspondence

$$\begin{tikzcd}
C \arrow[r, phantom, "\subseteq"] \arrow[dr, "\text{finite surj.}"'] & X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\
 & X &
\end{tikzcd}$$

define $A^p(C;K_n^M): A^p(Y;K_n^M)\to A^p(X;K_n^M)$

$$\alpha\mapsto p_{1*}(p_2^*(\alpha)\cdot C), \qquad C\in A_C^{d_Y}(X\times Y;K^M_{d_Y})$$

The homotopy invariance comes from Prop 8.1. □

Prop 8.10 The $A^0(-,K_n^M)=K_n^M(-)$ is a Nisnevich sheaf with transfers and homotopy invariant.