Lecture 16

9 March 2022

Extremal length (cont.).

For any 1-chain I (∇)

$$L_I(p) := \sum_{xy \in E(p)} r(x,y)\,|I(x,y)|$$

$$L_I(P) := \inf_{p \in P} L_I(p).$$

$$W(I) = \sum_{xy \in E} r(x,y)\,\big(I(x,y)\big)^2$$

$$\lambda(P) := \sup_{\substack{\text{non-trivial} \\ 1\text{-chain } I}} \frac{L_I^2(P)}{W(I)}$$

$$(\lambda(P))^{-1} = \inf_{1\text{-chain } I} \frac{W(I)}{L_I^2(P)}$$

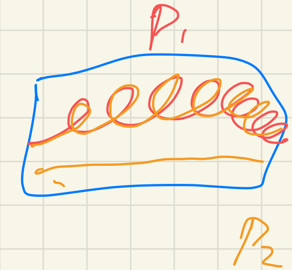
$$= \inf_{\substack{1\text{-chain } I \text{ s.t.} \\ L_I^2(P) \geq 1}} W(I)$$

Goal :

① Claim: $R_{eff} \sim \lambda$

Lemma: (I). $P_1 \subset P_2$

$$\lambda(P_1) \geq \lambda(P_2)$$

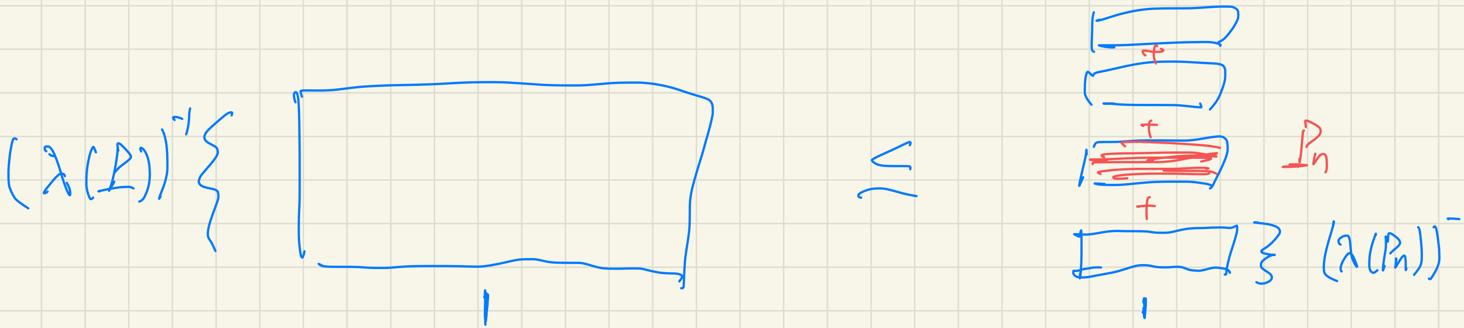


Proof: For any I, $L_I(P_1) \geq L_I(P_2)$

$$\lambda(P_1) = \sup_I \frac{L_I^2(P_1)}{w(I)} \geq \sup_I \frac{L_I^2(P_2)}{w(I)} = \lambda(P_2)$$

(II). $\{P_n\}$ countable family of paths and $P = \bigcup_{n=1}^{\infty} P_n$

then $$(\lambda(P))^{-1} \leq \sum_{n=1}^{\infty} (\lambda(P_n))^{-1}$$



<u>Pf:</u> For each n, $\exists$ I_n s.t.

$$L_{I_n}(P_n) \ge 1$$

and

$$W(I_n) < (\lambda(P_n))^{-1} + \frac{\varepsilon}{2^n}$$

For each edge x, y,

$$I(x,y) := \sup_n I_n(x,y) < \infty$$

$$\Rightarrow \quad L_I(P_n) \geq 1 \quad \text{for all } n$$

$$\Rightarrow \quad L_I\left(\bigcup_{n=1}^{\infty} P_n\right) \geq 1.$$

$$(\lambda(P))^{-1} \leq w(I) \leq \sum_{n=1}^{\infty} w(I_n) < \sum_{n=1}^{\infty}\left((\lambda(P_n))^{-1} + \frac{\varepsilon}{2^n}\right)$$

$$= \varepsilon + \sum_{n=1}^{\infty} (\lambda(P_n))^{-1}$$

Since $\varepsilon > 0$ is arbitrary,

$$\Rightarrow \quad (\lambda(P))^{-1} \leq \sum_{n=1}^{\infty} (\lambda(P_n))^{-1}$$

(II) $\lambda(\mathcal{P}) = \infty$ if and only if $\exists\, I \in H$, s.t. $L_I(\mathcal{P}) = \infty$

$\Updownarrow$ $W(I) < \infty$.

$\Leftrightarrow$ $\displaystyle\sum_{xy \in E(p)} r(x,y)\,|I(x,y)| = \infty$ for all paths in $\mathcal{P}$.

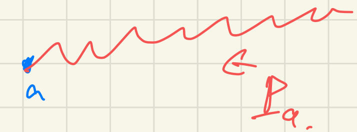
Pf:

$$\lambda(\mathcal{P}) = \sup_{I \text{ non-trivial}} \frac{(L_I(\mathcal{P}))^2}{W(I)}$$

$\lambda(\mathcal{P}) = \infty \iff \exists\, I$ s.t. $0 \neq W(I) < \infty$ and $L_I(\mathcal{P}) \to \infty$.

$\checkmark$

Def. Pick $a, b \in V$.



$$\mathcal{P}_{a,\infty} = \mathcal{P}_a := \{ \text{infinite one-sided paths starting at } a \}.$$

$$\mathcal{P}_{a,b} = \{ \text{finite paths starting at } a \text{ and ending at } b \}.$$

Thm (Γ, r) finite network.

Then $$R_{eff}(a,b) = \lambda(\mathcal{P}_{a,b})$$

Pf $(\le)$

Pick $p \in \mathcal{P}_{a,b}$. Denote $p = \{ \overset{a}{\overset{\shortparallel}{x_0}}, x_1, x_2, \dots, \overset{b}{\overset{\shortparallel}{x_n}} \}$.

Let $u : V \to \mathbb{R}$ s.t. $u(a) = 1$, $u(b) = 0$.

$\Rightarrow$ $I(x,y) := c(x,y)\left(u(x) - u(y)\right)$.

$W(I) = D(u)$.

Show: $L_I(p) \geq 1$

$$L_I(p) = \sum_{xy \in \overline{E}(p)} r(x,y)\, c(x,y)\, \left|u(x) - u(y)\right|$$

$$= \sum_{xy \in \overline{E}(p)} \left|u(x) - u(y)\right|$$

$$\geq \sum_{xy \in \overline{E}(p)} u(x) - u(y) = u(a) - u(b)$$

$$= 1$$

$$(\lambda(\mathcal{P}))^{-1} = \inf_{\substack{I \text{ s.t.} \\ L_I(\mathcal{P}) \geq 1}} W(I) \leq D(u) \quad \text{for all } u \text{ s.t. } u(a)=1,\ u(b)=0$$

$$\Rightarrow \quad (\lambda(\mathcal{P}))^{-1} \leq \inf\{D(u) \mid u(a)=1,\ u(b)=0\},$$

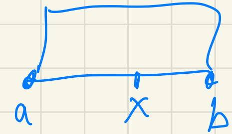
$$= (R_{\text{eff}}(a,b))^{-1}$$

$$\Rightarrow \quad R_{\text{eff}}(a,b) \leq \lambda(\mathcal{P})$$

Conversely, let 1-chain I s.t. $L_I(\mathcal{P}_{a,b}) \geq 1$

and $W(I) \leq (\lambda(\mathcal{P}_{a,b}))^{-1} + \varepsilon,$

We want to define "voltage".



$$u(a) = 0.$$

$$u(x) = \inf_{\substack{p \text{ is a path} \\ \text{from } a \text{ to } x.}} \left(\sum_{xy \in E(p)} r(x,y) \, |I(x,y)| \right)$$

Suppose y is adjacent to x.

$$u(y) \leq u(x) + r(x,y) \, I(x,y)$$

$$u(x) \leq u(y) + r(x,y) \, I(x,y)$$

$$\Rightarrow \quad | u(x) - u(y) | \leq r(x,y) \, I(x,y).$$

$$\Rightarrow \quad \frac{1}{r(x,y)} |u(x)-u(y)|^2 \leq r(x,y) \cdot (I(x,y))^2$$

$$\Rightarrow \quad D(u) \leq W(I) \leq [\lambda(P_{a,b})]^{-1} + \varepsilon,$$

Notice $\quad u(b) = L_I(P_{a,b}) = 1$

$\quad u(a) = 0$

$$\Rightarrow \quad [R_{eff}(a,b)]^{-1} = \inf \{ D(\hat{u}) \mid \hat{u}(b)=1, \hat{u}(a)=0 \}$$

$$\leq D(u)$$

$$\leq (\lambda(P_{a,b})^{-1}) + \varepsilon.$$

$$\Rightarrow \quad \lambda(\mathcal{D}_{a,b}) \leq R_{\mathrm{eff}}(a,b)$$

$$\Rightarrow \quad \lambda(\mathcal{D}_{a,b}) = R_{\mathrm{eff}}(a,b).$$

<u>Thm</u> (Γ, r) infinite network.

Then $R_{\mathrm{eff}}(a) = R_{\mathrm{eff}}(a,\infty) = \lambda(\mathcal{D}_{a,\infty})$.

Pf: (Γ_n, r_n) sequence of finite network.

$(\Gamma_n', r_n') \Leftrightarrow V_n' = V_n \cup \{b_n\}$.

$$R_{eff}(a, b_n) = \lambda(P_{a,b_n}),$$

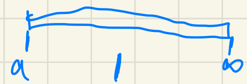
$$\downarrow \text{ as } n \to \infty \qquad \downarrow \text{ as } n \to \infty$$

$$R_{eff}(a, \infty) \qquad \lambda(P_a)$$

Exercise.

<u>Corollary</u> (Γ, r) is recurrent $\Leftrightarrow$ $\exists\, a \in V$ s.t. $\lambda(P_a) = \infty$

$$\left(\begin{array}{c} \| \\ R_{eff}(a, \infty). \end{array} \right)$$

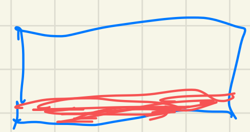
Recurrent: $0 = \frac{1}{\lambda}$ {  $\longleftrightarrow$ height $= \frac{1}{\lambda}$ somehow measures how large is $P_{a,b}$.

<u>Def</u> $\underline{P}$ set of paths, $(\lambda(P) < \infty)$

We say a property holds for almost every path in $\underline{P}$,

if the subset $P_0 \subset \underline{P}$ consisting of paths where

the property does not hold has $\lambda(P_0) = \infty$.

$0 = \frac{1}{\lambda(P_0)}$ {  P_0