$H^{p,1}(X,\mathbb{Z})$, $\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)=\{f\in k(X\times\mathbb{P}^1) \mid f|_{\{0,\infty\}}=1\}$.

Prop 7.2 Suppose $X\in Sm/k$, $C\in Cor_k(X,\mathbb{G}_m)$ (integral), then C is a principal divisor on $X\times\mathbb{G}_m$. Suppose $C=\operatorname{div} f|_{X\times\mathbb{G}_m}$ where $f\in k(X\times\mathbb{P}^1)$. Then f can be chosen to be the form $t^n+a_1t^{n-1}+\cdots a_n$, where t is the parameter in $\mathbb{G}_m$, $a_1,\cdots,a_n\in\mathcal{O}_X(X)$ and $n\in[k(C):k(X)]$. In this case $\operatorname{supp}(\operatorname{div}(f))\cap\{X\times\{0\}\}=\phi$ and $a_n\in\mathcal{O}_X^*(X)$.

Proof. For every affine open set $U\subseteq X$, we define $f_U\in\mathcal{O}(U)[t]$ to be the minimal polynomial of $t|_C$ over $k(U)$, it has degree $[k(C):k(X)]$. These coefficients glues together so we obtain our $f=t^n+a_1t^{n-1}+\cdots+a_n$. $a_n\in\mathcal{O}^*$ because it locally does. We have $\operatorname{div} f=C$ since f is the minimal polynomial of t and $f|_C=0$. Also $\operatorname{supp}(\operatorname{div}(f))\cap\{X\times 0\}=\phi$ because $a_n\in\mathcal{O}^*(X)$. □

Prop 7.3 Suppose $X\in Sm/k$, $f\in k(X\times\mathbb{P}^1)$, $\operatorname{supp}(\operatorname{div} f)\subseteq X\times\mathbb{G}_m$ and $\operatorname{supp}(\operatorname{div} f)$ $\in Cor_k(X,\mathbb{G}_m)$. Then

$$\mathcal{O}^*(\operatorname{div} f)(t)=\frac{f|_{t=0}}{f|_{t=\infty}}\in\mathcal{O}^*(X)$$

where $t\in\mathcal{O}^*(\mathbb{G}_m)$ is the parameter.

Proof. We could check locally so we assume X is affine. Suppose $\operatorname{div}(f)=\sum_a n_a\cdot C_a$ and $g_a\in\mathcal{O}(X)[t]$ be the polynomial corresponding C_a in Prop 7.2. Then

$$\mathcal{O}^*(\operatorname{div} f)(t)=\prod_a\left((-1)^{\deg g_a}g_a(0)\right)^{n_a} \qquad (A[t]_t)^*=A^*\oplus\mathbb{Z}$$

by basic facts of norms. Since $\operatorname{div}(g_a)=C_a$, $\frac{f}{\prod_a g_a^{n_a}}\in\mathcal{O}^*(X\times\mathbb{G}_m)$. So $\frac{f}{\prod_a g_a^{n_a}}=u\cdot t^m$, where $u\in\mathcal{O}^*(X)$, $m\in\mathbb{Z}$. But $\operatorname{supp}(\operatorname{div} f)\cap(X\times 0)=\phi$, so $m=0$. So $f=u\cdot\prod_a g_a^{n_a}=u\cdot t^{\sum n_a\deg g_a}\cdot\prod_a\left(\frac{g_a}{t^{\deg g_a}}\right)^{n_a}$. So $\operatorname{ord}_{X\times\infty}(f)=-\sum_a n_a\cdot\deg g_a=0$ since $\operatorname{supp}(\operatorname{div} f)\cap(X\times\infty)=\phi$.

So $f|_{t=0}=u\cdot\prod_a g_a(0)^{n_a}$, $f|_{t=\infty}=u$. Hence

$$\mathcal{O}^*(\operatorname{div} f)(t)=\prod_a g_a(0)^{n_a}=\frac{f|_{t=0}}{f|_{t=\infty}}.\ \square$$

$\boxed{\pi:\mathbb{G}_m\to\operatorname{Spec}k}$

Prop 7.4 $\forall X\in Sm/k$, there is an exact sequence of abelian groups.

$$0\to\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)\xrightarrow{\operatorname{div}}Cor_k(X,\mathbb{G}_m)\xrightarrow{\lambda}Cor_k(X,\operatorname{Spec}k)\oplus\mathcal{O}^*(X)\to 0$$

where $\lambda(C)=(\pi\circ C,\mathcal{O}^*(C)(t))$.

Proof: i) div is injective

Suppose $\operatorname{div} f=0$, then $f\in\mathcal{O}^*(X\times\mathbb{P}^1)$, since $\mathcal{O}(X\times\mathbb{P}^1)=\mathcal{O}(X)$, so f comes from $\mathcal{O}^*(X)$. But $f|_{t=0}=1$, so $f=1$ as well.

ii) $\lambda\circ\operatorname{div}=0$.

We have a commutative diagram

$$\begin{tikzcd} X\times\mathbb{G}_m \arrow[r, "a"] \arrow[d, hook, "i"] & X \\ X\times\mathbb{P}^1 \arrow[ur, "b"'] & \end{tikzcd}$$

Then

$\pi\circ(\operatorname{div} f)=a_*(\operatorname{div} f)=b_*(\operatorname{div} f)\in CH^0(X)$

Base changing to $\operatorname{Spec}k(X)$, we find that $b_*(\operatorname{div} f)=\deg_{\mathbb{P}^1_{k(X)}}(\operatorname{div}(f_{k(X)}))=0$.

Moreover, $\mathcal{O}^*(\operatorname{div} f)(t)=\frac{f|_{t=0}}{f|_{t=\infty}}=1$ by Prop 7.3.

iii) $\ker(\lambda)\subseteq\operatorname{Im}(\operatorname{div})$. $\quad p_1:X\times\mathbb{G}_m\to X$

Suppose $\sum_a n_aC_a\in Cor_k(X,\mathbb{G}_m)$ satisfies $p_{1*}(\sum n_a\cdot C_a)=0$ and $\mathcal{O}^*(\sum n_a\cdot C_a)(t)=1$.

$\forall C_a$, pick the $f_a\in\mathcal{O}(X)[t]$ given by Prop 7.2. So We have $(\operatorname{supp}(\operatorname{div}\prod_a f_a^{n_a}))\cap X\times 0=\phi$.

Moreover $p_{1*}(\sum n_a\cdot C_a)=\sum n_a\cdot\deg f_a\cdot p_1(C_a)=0$. So $\sum_a n_a\cdot\deg f_a=0$.

Suppose $f_a=t^d+a_1t^{d-1}+\cdots+a_d=t^d(1+\cdots)$. So $\prod_a f_a^{n_a}=\prod_a h_a^{n_a}$ where $h_a\in\mathcal{O}(X\times\mathbb{P}^1\setminus 0)$ and $h_a|_{t=\infty}=1$. Hence $\prod_a f_a^{n_a}|_{t=\infty}=1$, so $(\operatorname{supp}\operatorname{div}\prod_a f_a^{n_a})\cap(X\times\infty)=\phi$. By Prop 7.3, we have

$$1=\mathcal{O}^*(\sum n_a\cdot C_a)(t)=\mathcal{O}^*(\operatorname{div}\prod_a f_a^{n_a})(t)=\frac{\prod_a f_a^{n_a}|_{t=0}}{\prod_a f_a^{n_a}|_{t=\infty}},$$

so $\prod_a f_a^{n_a}|_{t=0}=1$ as well.

So $\prod_a f_a^{n_a}\in\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)$ and $\operatorname{div}(\prod_a f_a^{n_a})=\sum n_a f_a$.

iv) λ is surjective

Denote by $\beta:\operatorname{Spec}k\to\mathbb{G}_m$ the map constantly equal to 1. Then for every $C\in Cor_k(X,\operatorname{Spec}k)$, $\pi\circ\beta\circ C=C$ and $\mathcal{O}^*(\beta\circ C)(t)=1$. So $C(1,1)=\lambda(\beta\circ C)$. On the other hand, for any $u\in\mathcal{O}^*(X)$, it corresponds to a $\varphi:X\to\mathbb{G}_m$. Hence $\lambda(\varphi)=(\pi\circ\varphi,u)$. So λ is surjective. □

Prop 7.5 Let A be a simplicial object in Ab, namely a functor $A:\operatorname{Sm}^{op}\to Ab$ ($\to[0]\cdots[n]\cdots$). Define the normalized complex $C_*^{DK}A\subseteq C_*A$ by $(C_*^{DK}A)_n=\{x\in A_n \mid \partial_i(x)=0,\forall i<n\}$ (Dold-Kan). Then $C_*^{DK}A$ is quasi-isomorphic to C_*A.

Proof: [Theorem 2.4, Goerss, Jardine, Simplicial Homotopy Theory]. □

Prop 7.6 $\forall X\in Sm/k$, the $(C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X)$ is an acyclic complex.

Proof: Denote by $i_0,i_1:X\to X\times\mathbb{A}^1$, $x\mapsto\begin{cases}(x,0)\\(x,1)\end{cases}$ respectively. By Lem 5.17 the two maps

$i_0^*,i_1^*:(C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X\times\mathbb{A}^1)\to((C_*\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X)$ are chain homotopic so $H_*(i_0^*)=H_*(i_1^*)$. By Prop 7.5, $H_*(i_0^*)=H_*(i_1^*)$ also holds for normalized complexes. Suppose $f\in Z_n((C_*^{DK}\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X)\subseteq\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X\times\sigma^n)\subseteq k(X\times\sigma^n\times\mathbb{P}^1)$. Define $g=1-t(1-f)\in k(X\times\mathbb{A}^1\times\sigma^n\times\mathbb{P}^1)$, where t is the parameter in $\mathbb{A}^1$. Then $g|_{X\times\mathbb{A}^1\times\sigma^n\times\{0,\infty\}}=1$ so $g\in\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X\times\mathbb{A}^1\times\sigma^n)$
$=(C_n\mathcal{M}^*(\mathbb{P}^1;0,\infty))(X\times\mathbb{A}^1)$.

The $g|_{X\times\mathbb{A}^1\times\sigma^{n-1}\times\mathbb{P}^1}=1$ for any face $\sigma^{n-1}\subseteq\sigma^n$ because f does. Moreover $g|_{X\times 0\times\sigma^n\times\mathbb{P}^1}=1$ and $g|_{X\times 1\times\sigma^n\times\mathbb{P}^1}=f$. So f is a boundary. □

Thm 7.7 The map $\mathbb{Z}(\mathbb{G}_m^{\wedge 1})\xrightarrow{\bar\lambda}\mathcal{O}^*$ is an $\mathbb{A}^1$-weak equivalence.

Proof: By Prop 7.4, we have an exact sequence

$$0\to C_*(\mathcal{M}^*(\mathbb{P}^1;0,\infty))\to C_*\mathbb{Z}(\mathbb{G}_m^{\wedge 1})\xrightarrow{C_*\bar\lambda}C_*\mathcal{O}^*\to 0.$$

So by Prop 7.6, the $C_*\bar\lambda$ is a quasi-isomorphism. Hence the statement follows by Prop 5.31. □ $\quad \mathbb{Z}(\mathbb{G}_m^{\wedge 1})\to\mathcal{O}^*$

Prop 7.8 If $F\in Sh(k)$ is homotopy invariant. Then

$$H^i_{Zar}(X,F)=H^i_{Nis}(X,F)$$

for every $i\in\mathbb{N}$, $X\in Sm/k$.

Proof: We have a functor $\pi:Sh_{Nis}\to Sh_{Zar}$ and Leray spectral sequence

$$H^p_{Zar}(X,R^q\pi_*F)\Rightarrow H^{p+q}_{Nis}(X,F).$$

So it suffices to show that $R^q\pi_*F=0$ if $q>0$. The $R^q\pi_*F$ is the Zariski sheafification of the presheaf $X\mapsto H^q_{Nis}(X,F)$. But it's a presheaf with transfers (h.i.) whose sections at field vanishes. So by Thm 6.7 we conclude

Cor 7.9 We have $H^{p,q}(X,\mathbb{Z})=\mathbb{H}^p_{Zar}(X,\mathbb{Z}(q))$, if k is perfect.

Proof: The homology sheaves of $\mathbb{Z}(q)$ are homotopy invariant by Thm 5.36. So the result follows from hypercohomology spectral sequence and Prop 7.8. □

Prop 7.10 We have $H^{p,1}(X,\mathbb{Z})=\begin{cases}\mathcal{O}^*(X) & p=1\\ \operatorname{Pic}(X)=CH^1(X) & p=2\\ 0 & \text{else.}\end{cases}$

Proof: By Thm 7.7 and Prop 7.8, $H^{p,1}(X,\mathbb{Z})=H^{p-1}_{Nis}(X,\mathcal{O}^*)=H^{p-1}_{Zar}(X,\mathcal{O}^*)$ ($\mathbb{Z}(\mathbb{G}_m^{\wedge 1})\cong\mathcal{O}^*$; $\mathbb{Z}(1)[1]$; $\mathcal{O}^*$-h.i.). So the statement for $p\leq 2$ follows. On the other hand, we have an exact sequence $0\to\mathcal{O}^*\to K^*\xrightarrow{\operatorname{div}}\bigoplus_{x\in X^{(1)}}\mathbb{Z}\to 0$ since X is smooth. This is a flasque resolution of $\mathcal{O}^*$ hen $H^i(X,\mathcal{O}^*)=0$ if $i>1$. □

§8 Comparison Theorem, IV.

In this section we want to compute $H^{p,q}(X,\mathbb{Z})$ if $p\geq 2q-1$.

Let us write $A^p(X;K_n^M)=H^p(C^*(X;K_n^M))$. By Prop 3.17, for any

$$(A^0(X;K_n^M)=K_n^M(X),\ A^n(X;K_n^M)=CH^n(X))$$

flat morphism $f:X\to Y$, we have pull-back $f^*:A^p(Y;K_n^M)\to A^p(X;K_n^M)$. Moreover, every proper morphism $g:X\to Y$ induces a push-forward

$$g_*:A^p(X;K_n^M)\to A^{p+d_Y-d_X}(Y;K^M_{n+d_Y-d_X})?$$