$$\mathrm{Hom}_{DM}(K,L) \overset{\otimes\mathbb{Z}(1)}{=} \mathrm{Hom}_{DM}(K(1),L(1))$$

**Prop 6.5** Let $F\in Sh(k)$ s.t. there is an epimorphism $\mathbb{Z}(X)\to F$ for some $X\in Sm/k$. Let $\varphi: \mathbb{G}_m^{\wedge 1}\otimes F \to \mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y)$ be a map of sheaves. Then there exists a unique up to an $\mathbb{A}^1$-homotopy morphism $\rho(\varphi): F\to\mathbb{Z}(Y)$ s.t. $\mathbb{G}_m^{\wedge 1}\otimes\rho(\varphi)\overset{\mathbb{A}^1}{\sim}\varphi$
($\overset{\mathbb{A}^1}{\sim}$, $\mathrm{Hom}(F,G)/\langle\mathbb{A}^1\text{-homotopies}\rangle$).

Proof: Let us fix an epimorphism $p:\mathbb{Z}(X)\to F$. Then the morphism ($\mathbb{G}_m=\mathbb{G}_m^{\wedge 1}\oplus\mathbb{Z}$) $\varphi_+\circ(Id_{\mathbb{G}_m}\times p)$ gives a $Z\in Cor_k(\mathbb{G}_m\times X,\mathbb{G}_m\times Y)$ and for sufficiently large $n$ ($\varphi_+=\varphi\oplus id_{pt}: \mathbb{G}_m\otimes F\to\mathbb{G}_m\otimes\mathbb{Z}(Y)$) we may consider $\rho_n(Z): X\to Y$. Suppose $f: W\to X$ is a finite correspondence s.t. $p\circ f=0$, then by Prop 6.3, we have
$$\rho_n(Z)\circ f=\rho_n(Z_n\circ(Id_{\mathbb{G}_m}\times f))=\rho_n(\varphi_+\circ(Id_{\mathbb{G}_m}\times p)\circ(Id_{\mathbb{G}_m}\times f))=0.$$
So $\rho_n(Z)|_{\ker(p)}=0$, hence we obtain a morphism $\rho_n(\varphi): F\to\mathbb{Z}(Y)$.

Let us show that for a sufficient large $n$ one has $\mathbb{G}_m^{\wedge 1}\otimes\rho_n(\varphi)\overset{\mathbb{A}^1}{\sim}\varphi$.
Define $\varphi'$ by the diagram
$$\begin{array}{ccc} \mathbb{G}_m^{\wedge 1}\otimes F & \xrightarrow{\varphi} & \mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y). \\ \downarrow\wr & & \downarrow\wr \\ F\otimes\mathbb{G}_m^{\wedge 1} & \xrightarrow{\varphi'} & \mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1} \end{array}$$

We claim that $\varphi\otimes Id_{\mathbb{G}_m^{\wedge 1}}$ and $Id_{\mathbb{G}_m^{\wedge 1}}\otimes\varphi'$ are $\mathbb{A}^1$-homotopic.
$(t_1,s,t_2)\mapsto(\varphi(t_1,s),t_2)$ $\qquad (t_1,s,t_2)\mapsto(t_1,\varphi(s,t_2))$
These morphisms differs by a conjugation of the swaping map of $\mathbb{G}_m^{\wedge 2}$. $\qquad (t_1,\varphi(t,s))$

Lemma: The swaping map is $\mathbb{A}^1$-homotopic to the map
$$\mathbb{G}_m^{\wedge 2}\to\mathbb{G}_m^{\wedge 2}$$
$$(x,y)\mapsto(x,y^{-1})$$
which proves the claim before.

Proof of the lemma: For any $f_1,\cdots,f_n: X\to\mathbb{G}_m$, we write $\tilde f_1,\cdots,\tilde f_n: X\to\mathbb{G}_m^{\wedge 1}$ by composing the projection $\mathbb{G}_m\to\mathbb{G}_m^{\wedge 1}$. Then we write $\tilde f_1\otimes\cdots\otimes\tilde f_n: X\to\mathbb{G}_m^{\wedge n}$ as $[f_1]\cdots[f_n]$. ($\tilde f_i\neq 1$ anywhere)
Given $f_1,f_2,g: X\to\mathbb{G}_m$, define the correspondence $Z\in Cor_k(X\times\mathbb{A}^1,\mathbb{G}_m)$ by $(x,t,y)\in X\times\mathbb{A}^1\times\mathbb{G}_m$,
$$y^2-\left(t(\tilde f_1(x)+\tilde f_2(x))+(1-t)(1+\tilde f_1(x)\tilde f_2(x))\right)y+\tilde f_1(x)\tilde f_2(x)=0.$$
We have $t=\dfrac{y^2+t\tilde f_1\tilde f_2 y+\tilde f_1\tilde f_2}{y(\tilde f_1-1)(\tilde f_2-1)}$ so $Z=X\times\mathbb{G}_m$. The projection $Z\to X\times\mathbb{A}^1$ is finite by locally checking. We have $Z|_{t=0}=[1]+[f_1][f_2]$, $Z|_{t=1}=[f_1]+[f_2]$,
so $[1]+[f_1][f_2]\overset{\mathbb{A}^1}{\sim}[f_1]+[f_2]$, $[f_1f_2][g]\overset{\mathbb{A}^1}{\sim}[f_1][g]+[f_2][g]$.
$[f_1]\overset{\shortparallel}{}[f_2]$

On the other hand, consider $Z\in Cor_k(X\times\mathbb{A}^1,\mathbb{G}_m\times\mathbb{G}_m)$ ($(x,t,y_1,y_2)\in X\times\mathbb{A}^1\times\mathbb{G}_m\times\mathbb{G}_m$) given by ($f,g: X\to\mathbb{G}_m$)
$$\begin{cases} y_1^2-\left(t(\tilde f(x)+\tilde g(x))+(1-t)(1+\tilde f(x)\tilde g(x))\right)y_1+\tilde f(x)\tilde g(x)=0 \\ y_1=y_2 \end{cases},$$
which gives the relation $[fg][fg]\overset{\mathbb{A}^1}{\sim}[f][f]+[g][g]$.
But $[fg][fg]\overset{\mathbb{A}^1}{\sim}[f][f]+[f][g]+[g][f]+[g][g]$, so $[f][g]+[g][f]\overset{\mathbb{A}^1}{\sim}0$.
Then $[g][f]+[g][f^{-1}]\overset{\mathbb{A}^1}{\sim}[g][1]=0$ implies that $[f][g]\overset{\mathbb{A}^1}{\sim}[g][f^{-1}]$.
So the identity map of $\mathbb{G}_m^{\wedge 2}$ is $\mathbb{A}^1$-homotopic to $(x,y)\mapsto(y,x^{-1})$, hence proves the statement. □
$X\underset{g}{\overset{f}{\rightrightarrows}}\mathbb{G}_m^{\wedge 1}$ $\quad X\xrightarrow[(g,f^{-1})]{(f,g)}\mathbb{G}_m^{\wedge 2}$ $\quad (x,y)\mapsto(y,x)$ $\quad (x,y^{-1})$

For sufficiently large $n$, we have
$$\rho_n(\varphi\otimes Id_{\mathbb{G}_m^{\wedge 1}})=\rho_n(\varphi)\otimes Id_{\mathbb{G}_m^{\wedge 1}}$$
by Prop 6.4. On the other hand,
$$\rho_n(Id_{\mathbb{G}_m^{\wedge 1}}\otimes\varphi')=\varphi'$$
by Prop 6.2. Hence $\varphi'=\rho_n(Id_{\mathbb{G}_m^{\wedge 1}}\otimes\varphi')\overset{\mathbb{A}^1}{\sim}\rho_n(\varphi\otimes Id_{\mathbb{G}_m^{\wedge 1}})=\rho_n(\varphi)\otimes Id_{\mathbb{G}_m^{\wedge 1}}$ by the claim. So $Id_{\mathbb{G}_m^{\wedge 1}}\otimes\rho_n(\varphi)\overset{\mathbb{A}^1}{\sim}\varphi$. This prove the existence of $\rho(\varphi)$.

For the uniqueness, consider a morphism $\varphi$ of the form $Id_{\mathbb{G}_m^{\wedge 1}}\otimes\psi$. Then the $Z\in Cor_k(\mathbb{G}_m\times X,\mathbb{G}_m\times Y)$ defined before is of the form $Id_{\mathbb{G}_m^{\wedge 1}}\otimes w$ where $w\in Cor_k(X,Y)$ corresponds to $\psi$. By Prop 6.2, we $\rho_n(Z)=w$. If $\rho,\rho'$ satisfy $Id_{\mathbb{G}_m^{\wedge 1}}\otimes\rho\overset{\mathbb{A}^1}{\sim}\varphi\overset{\mathbb{A}^1}{\sim}Id_{\mathbb{G}_m^{\wedge 1}}\otimes\rho'$ then applying $\rho_n$ for large $n$ we find $\rho\overset{\mathbb{A}^1}{\sim}\rho'$. □

**Prop 6.6** Denote by $F_Y$ the presheaf
$$F_Y(X)=\mathrm{Hom}(\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(X),\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y))$$
and consider the obvious map $\mathbb{G}_m^{\wedge 1}\otimes:\mathbb{Z}(Y)\to F_Y$, $f\mapsto\mathbb{G}_m^{\wedge 1}\otimes f$. Then for any $X\in Sm/k$, the map between complexes
$$\theta: C_*(\mathbb{Z}(Y))(X)\to C_*(F_Y)(X)$$
is a quasi-isomorphism.
Proof: The map $C_p(\mathbb{Z}(Y))(X)\to C_p(F_Y)(X)$ is just the map $\mathrm{Hom}(\mathbb{Z}(X\times\Delta^p),\mathbb{Z}(Y))\xrightarrow{\mathbb{G}_m^{\wedge 1}}\mathrm{Hom}(\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(X\times\Delta^p),\mathbb{G}_m^{\wedge 1}\otimes\mathbb{Z}(Y))$. For any $G$, we have
$$Z_p(C_*(G)(X))=\mathrm{Hom}(\mathbb{Z}(X)\otimes\mathbb{Z}(\Delta^p/d\Delta^p),G).$$

So by Prop 6.5, for every $f\in Z_p(C_*(F_Y)(X))$, there is a $g\in Z_p(C_*(\mathbb{Z}(Y))(X))$ s.t. $\theta(g)\overset{\mathbb{A}^1}{\sim}f$, so $\theta(g)-f\in B_p(C_*(F_Y)(X))$ by using the chain homotopy in Lemma 5.17
So $H_*(\theta)$ is surjective. $\qquad s_n: G^{\Delta^n}(X\times\mathbb{A}^1)\to G^{\Delta^{n+1}}(X)$
On the other hand, if $\theta(g)=d(f)$, then it's easy to check that $d(\rho_n(f))=g$ for large $n$. So $\theta$ is a quasi-isomorphism. □ (Prism decomposition of $\Delta^n\times\mathbb{A}^1$)

**Thm 6.7** Let $F\in PSh(k)$ be homotopy invariant and $F(\mathrm{Spec}\,E)=0$ for every field $E/k$, then $F=0$ after Zariski sheafification. ($f: F_1\to F_2$, $f(E)$ is iso $\Rightarrow$ $f$ iso)
Proof. See [MVW, Corollary 11.2]. □

$\boxed{p_*F=F^{\mathbb{G}_m}}$

**Prop 6.8** Let $F\in Sh(k)$, $k$ perfect, be homotopy invariant, then $R^ip_*F=0$, $\forall i>0$, where $p:\mathbb{G}_m\to\mathrm{Spec}\,k$ is the structure map.
Proof: The $R^ip_*F$ is the sheaf associated to the presheaf $X\mapsto H^i(X\times\mathbb{G}_m,F)$.
Suppose $X=\mathrm{Spec}\,E$ where $E$ is field. Then we have an exact sequence
$$\underbrace{H^i(\mathbb{A}^1_E,F)}_{=H^i(\mathrm{Spec}E,F)=0}\to H^i((\mathbb{G}_m)_E,F)\to \underbrace{H^{i+1}_0(\mathbb{A}^1_E,F)}_{=0}\quad(i>0).$$
Hence $H^i((\mathbb{G}_m)_E,F)=0$. But the presheaf defined above is also homotopy invariant with tranfers. So we see that $R^ip_*F=0$ if $i>0$ by Thm 6.7. □

**Thm 6.9** Suppose $k$ is perfect. For any $K,L\in DM^{eff,-}(k)$, the map
$$\mathrm{Hom}_{DM}(K,L)\xrightarrow{\otimes\mathbb{Z}(1)}\mathrm{Hom}_{DM}(K(1),L(1))$$
is an isomorphism.
Proof: By Lem 5.22, it suffices to prove that for any $X,Y\in Sm/k$, $n\in\mathbb{Z}$, we have
$$\mathrm{Hom}_{DM}(\mathbb{Z}(X),\mathbb{Z}(Y)[n])=\mathrm{Hom}_{DM}(\mathbb{Z}(X)\otimes\mathbb{G}_m^{\wedge 1},\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})$$
$$\shortparallel \qquad\qquad\qquad \shortparallel$$
$$H^n(X,C_*\mathbb{Z}(Y)) \qquad H^n(X\otimes\mathbb{G}_m^{\wedge 1},C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1}))$$
by $\mathbb{A}^1$-locality of $C_*$

By Prop 6.6, the map $C_*\mathbb{Z}(Y)\xrightarrow{\otimes\mathbb{G}_m^{\wedge 1}}C_*\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})$ is a quasi-isomorphism.
$$\shortparallel$$
$$\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1}))$$
We have a spectral sequence $H^j(X,R^ip_*(C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})))\Rightarrow H^{i+j}(X\times\mathbb{G}_m,C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1}))$. ($p:\mathbb{G}_m\to\mathrm{Spec}\,k$)
By the Lem 1.20, we have $R^ip_*(H^j(C_*\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1}))\Rightarrow R^{i+j}p_*(C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1}))$.

So by Prop 5.37 and Prop 6.8, we have $R^ip_*(C_*\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})=0$ if $i>0$ since $H^j(C_*F)=0$ if $j>0$ for any sheaf $F$. So we have
$$H^n(X,C_*\mathbb{Z}(Y))=H^n(X,\underline{\mathrm{Hom}}(\mathbb{G}_m^{\wedge 1},C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})))=H^n(X\otimes\mathbb{G}_m^{\wedge 1},C_*(\mathbb{Z}(Y)\otimes\mathbb{G}_m^{\wedge 1})).$$
□

**Rmk 6.10** The assumption that $k$ is perfect could be dropped by [F. Déglise, Integral Mixed motives in Equal Characteristic, Prop 8.1], which states that $DM^{eff,-}(k)\to DM^{eff,-}(k^{perf})$ is a fully faithful functor.

## §7. Comparison Theorem, II.

In this section, we want to identify $H^{p,1}(X,\mathbb{Z})$.

**Def 7.1.** Let $X\in Sm/k$. Define
$$\mathcal{M}^*(\mathbb{P}^1;0,\infty)(X)=\{f\in k(X\times\mathbb{P}^1)\mid f|_{X\times\{0,\infty\}}=1\}.$$