# Lecture 14 25 April 2022

$$HD = \left\{ u : V \to \mathbb{R} \;\middle|\; \sum_{y \in V(x)} c_{xy}(u_y - u_x) = 0 \quad \forall x \text{ and } u \text{ has finite energy } \|u\|_D < \infty \right\}$$

$= \{$ Harmonic Dirichlet functions $\}$

**Def** $u, \hat{u} : V \to \mathbb{R}$ $G$ $D$,

$$[u, \hat{u}] := \sum_{xy \in E} c_{xy} (u_y - u_x)(\hat{u}_y - \hat{u}_x)$$

$[\ ,\ ]$ semi-inner product, since $[e, e] = 0$. where $e$ is constant.

$[\ \ ] =$ Energy of $u$. $= D(u)$.

**Lemma** $HD$ is closed in $D$. $\left( \langle u, \tilde{u} \rangle = u(o)\tilde{u}(o) + [u, \tilde{u}] \right)$.

and $HD \perp D_0$ with respect to $[\ ,\ ]$.

Pf: ① Let $u \in HD$, $v \in D_0 \Rightarrow \exists\ v_n \in D$ with finite support

and $\| v_n - v \| \to 0$ as $n \to \infty$.

$$[u, v_n] = \sum_{xy \in E} c_{xy} (u_x - u_y)(v_{n,x} - v_{n,y})$$

$$= \sum_{x \in V} v_{n,x} \sum_{y \in V(x)} c_{xy} (u_x - u_y) = 0.$$

$$[u, v] = \lim_{n \to \infty} [u, v_n] = 0.$$

(2). Claim: $HD$ is closed

Suppose $\{u_n\} \subset HD$ s.t. $\{u_n\}$ converges to $u \in D$.

$\Rightarrow \lim_{n\to\infty} \| u_n - u \| = 0$

By Ex.

$\Rightarrow \lim_{n\to\infty} u_n(x) = u(x)$ for each $x \in V$.

$\Rightarrow 0 = \lim_{n\to\infty} \sum_{y \in V(x)} c_{xy} (u_{n,y} - u_{n,x}) = \sum c_{xy} (u_y - u_x)$

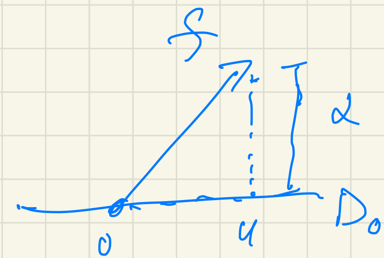
# Royden's decomposition Thm

$(\Gamma, r)$ transient network. For every $f \in D$, exists unique $\cancel{u}$ $v$ $\in HD$, $\cancel{v}$ $u$ $\in D_0$ s.t.

$$f = u + v \qquad \Rightarrow \quad D = HD \oplus D_0$$

and

$$D(f) = D(u) + D(v)$$




Pf:

$$\alpha := \inf_{u \in D_0} D(f-u)$$

$\Rightarrow$ $\exists$ $u_n \in D_0$ s.t.

$$D(f-u_n) < \alpha + \frac{1}{n} \qquad \forall n,$$

convexity of energy

$$\frac{1}{4} D(u_n) = D\left(\frac{u_n}{2}\right) = D\left(\frac{u_n - f + f}{2}\right) < \frac{1}{2} D(u_n - f) + \frac{1}{2} D(f)$$

$\Rightarrow \quad D(u_n) < 2D(u_n - f) + 2D(f) < 2\alpha + 2 + 2D(f)$

which holds for all $n$.

$\Rightarrow$ bounded sequence in $D$.

$\Rightarrow \exists\, u \in D$ s.t. $u_n$ converges weakly to $u$.

Exercise. $\quad D(f-u) \leq \liminf_{n\to\infty} D(f-u_n)$

( $D(\cdot)$ is lower semi-continuous ).

$\Rightarrow \quad \alpha = D(f-u)$

Check: $u \in D_0$.

Define $v := f - u$.

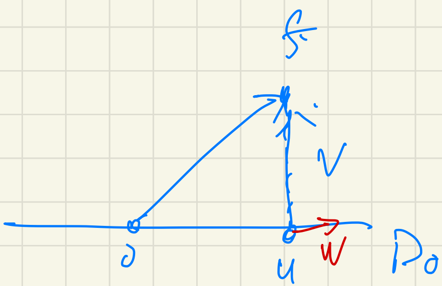
Claim: $v$ is harmonic.

Take any $w$ with finite support, For $t \in \mathbb{R}$,

$$D(v+tw) = D(v) + D(tw) + [v, tw]$$

$$\Rightarrow \quad D(v+tw) - D(v) = t^2 D(w) + t\,[v,w].$$

$$\Rightarrow \quad \frac{D(v+tw) - D(v)}{t} = t\,D(w) + [v,w]$$




Since $u \in D_0$ minimize $D(f-u)$ over $D_0$.

$$\Rightarrow \quad \frac{d}{dt} D(f-u+tw)\Big|_{t=0} = 0,$$

for all $w \in D_0$.

$$\Rightarrow \quad [v, w] = 0.$$

Consider $w = \delta_a = \begin{cases} 1 & \text{when } x = a \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow 0 = [v, \delta_a] = 1 \cdot \sum_{y \in V(a)} c_{ay} (v_y - v_a)$$

$\Rightarrow$ $v$ is harmonic.

For uniqueness, suppose $u, \hat{u} \in D_0$, $v, \hat{v} \in HD$, s.t.

$$f = u + v = \hat{u} + \hat{v}.$$

$$\Rightarrow \delta := \hat{u} - u = v - \hat{v} \in HD \cap D_0.$$

$[\delta, \delta] = 0 \Rightarrow \delta$ is constant.

$(\Gamma, r)$ is transient $\Rightarrow$ $D_0$ does not contain non-zero const. function

$\Rightarrow$ $\delta \equiv 0$.

$\Rightarrow$ $u = \hat{u}$, $v = \hat{v}$ $\Rightarrow$ decomposition is unique.

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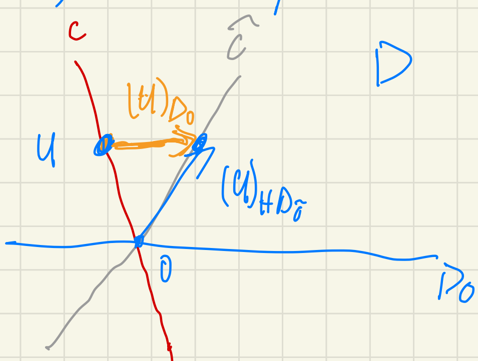
Remark: $D = HD \oplus D_0$ depends on edge weights (conductance)

$c, \hat{c}$ comparable, $\Rightarrow$ $D_{0,c} = D_{0,\hat{c}}$




$HD_c \neq HD_{\hat{c}}$

$\Leftrightarrow$ transversal subspace changes.

Take $u \in HD_c$.

$u \in D_c$

$c, \tilde{c}$ comparable $\Rightarrow$ $u \in D_{\tilde{c}}$

$\Updownarrow$ $\qquad \Rightarrow \quad u = (u)_{HD_{\tilde{c}}} + (u)_{D_0}$

$\exists K > 0$ s.t.

$$\frac{1}{K}\tilde{c}_{xy} < c_{xy} < K\tilde{c}_{xy}$$

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2.


Pick $h \in HD(\mathcal{U})$

$h$ classical smooth harmonic function over $\mathcal{U}_c$

Define: $g : V \to \mathbb{R}$ given by

$$g_i := h \circ z_i$$

Claim: If triangulation is nice,

$$D(g) \sim D(h) < \infty,$$

$$\Rightarrow \quad g = (g)_{HD(V)} + (g)_{D_0(V)}$$

$$\Rightarrow L: HD(\mathcal{U}) \longrightarrow HD(V)$$

$$h \longmapsto (h \circ z)_{HD(V)}$$

(Hutchcroft 2018) For circle packings, $L$ is isomorphic, and bounded linear operator.

Problem about $f \in HD$,


usually $\lim\limits_{n\to\infty} f(x_n) \to \infty$,

generally $f$ is unbounded and might converge to $\infty$ as one move to the boundary.

Approximate $HD$ by $BHD = \{$ bounded harmonic Dirichlet $\}$,

Goal: Show $BHD$ is dense in $HD$

**Lemma** Suppose $(\Gamma, r)$ transient and $u \in D_0$.

(I) Then positive and negative part of $u \in D_0$

If $u \in D_0$ is superharmonic, then

1), $u(x) \geq 0$ for all $x$.

2), $\exists f \geq 0$ s.t. $u = G(f)$.

Pf: $u \in D_0 \Rightarrow \exists u_n \in D_0$ with finite support and $\|u_n - u\| \to 0$ as $n \to \infty$.

$u^+(x) := \max\{u(x), 0\}$

$u^-(x) := \max\{-u(x), 0\}$

$\Rightarrow \quad u = u^+ - u^-$

(I) Ex: $\quad u^+, u^- \in D_0$

Idea: Show $\quad \|u_n^+ - u^+\| \to 0$ as $n \to \infty$

$\|u_n^- - u^-\| \to 0$ as $n \to \infty$.

1). Suppose $u \in D_0$ superharmonic. Want to show $u^- \equiv 0$.

$$|u| = u^+ + u^-$$

Claim: $\quad D(|u|) = D(u^+) = D(u)$

$$(\cancel{***}) \quad D(|u|) = \sum_{xy} c_{xy} \left| \underbrace{|u_x| - |u_y|}_{\leq |u_x - u_y|} \right|^2$$

$$\leq \sum_{xy} c_{xy} |u_x - u_y|^2 = D(u).$$

Note: $|u| = u^+ + u^- = u + 2u^-$

$$(\ast\ast\ast)\quad D(|u|) = D(u + 2u^-)$$

$$= D(u) + D(2u^-) + 4[u, u^-].$$

Take $\{u_n^-\}$ with finite support and converges to $u^-$.

$u \geq Pu$.

$$[u, u_n^-] = \sum_{x,y} c_{xy} (u_x - u_y)(u_{n,x}^- - u_{n,y}^-)$$

$\geq 0$ $\quad \geq 0$ since superharmonic

$$= \sum_{x \in V} u_{n,x}^- \left( \sum_{y \in V(x)} c_{xy} (u_x - u_y) \right)$$

$$\geq 0.$$

$\Rightarrow \quad (\ast) \quad D(|u|) \geq D(u).$

$\Rightarrow \quad D(|u|) = D(u).$

$(\ast\ast\ast) \Rightarrow \quad D(u^-) = 0$

$\Rightarrow \quad u^-$ is constant.

Since $u^- \in D_0$ and $(P, r)$ transient,

$\Rightarrow \quad u^- \equiv 0.$

$\Rightarrow \quad u \equiv u^+ \geq 0.$

Ex: (2). Show: $u \in D_0$ and superharmonic $\Rightarrow$ $u = G(f)$ for some $f \geq 0$.