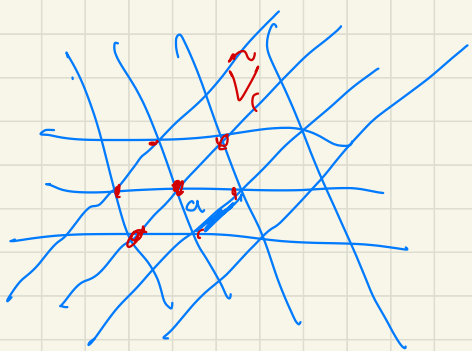
Lecture 13 24 April

1. Examples from circle packings
2. Royden's decomposition of D.

Eg. Triangular lattice in $\mathbb{R}^2$ is recurrent.



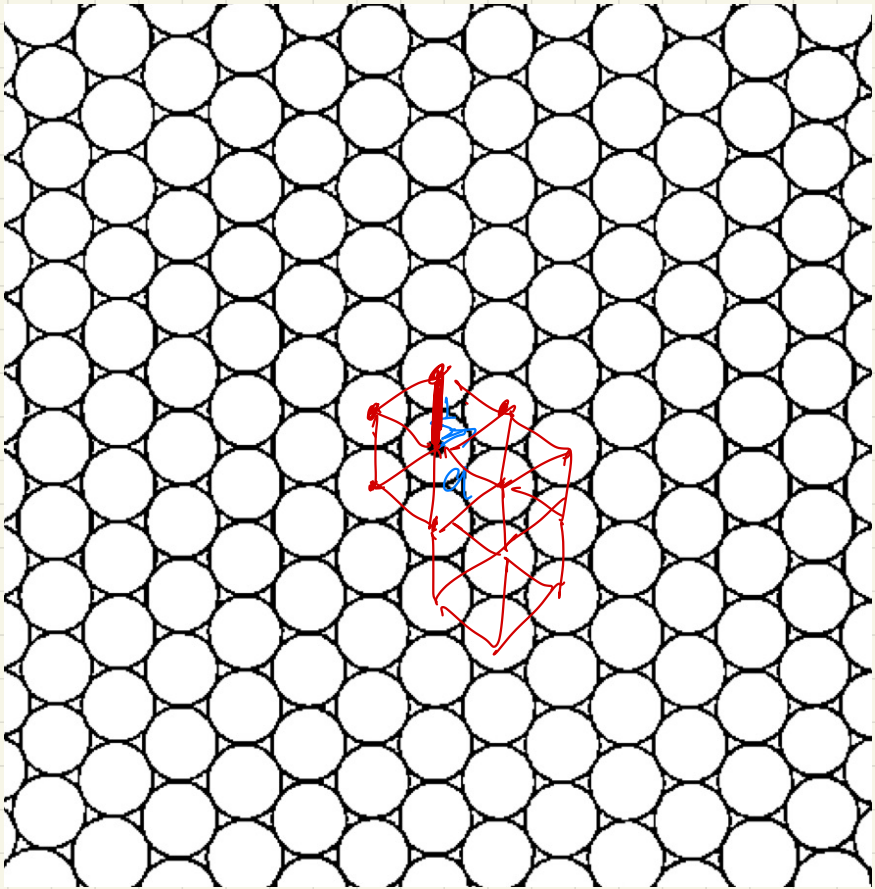
$$e_n := \#\{xy \in Y \mid x, y \in \tilde{V}_n\}$$

$$\leq \text{Const. } n^2$$

$$\Rightarrow R_{eff}(a, \infty) = \infty \Rightarrow \text{recurrent}$$

$$\tilde{V}_n := \{x \in V \mid d(x,a) \leq n\} \Rightarrow V_n := \tilde{V}_n - \tilde{V}_{n-1} \Rightarrow \{V_n\} \text{ canonical decomposition}$$

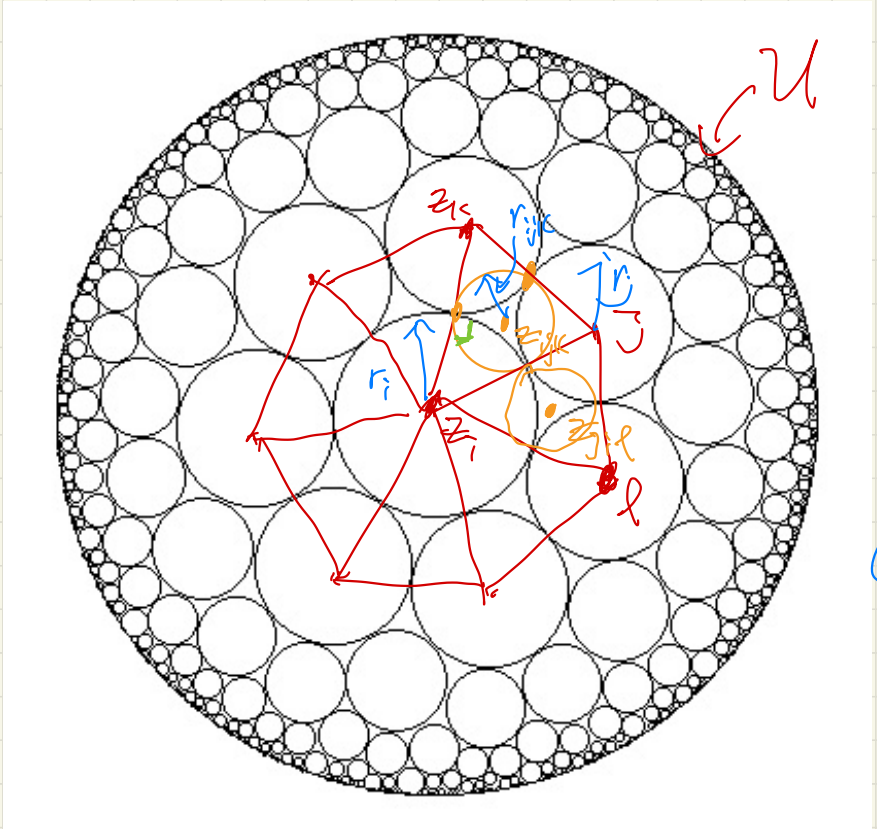
Densest circle packing → Triangular lattice



Claim: ① Suppose $G=(V,E)$ is contact graph of a circle packing that fills up a bounded domain $\mathcal{U}$, then G is transient.

② no circle touches the boundary

Here, boundary is the accumulation points of circles



$X: V \to \mathbb{R}$

harmonic

Remark:

① unit circle is boundary of the graph.

② harmonic functions have well-defined value along the boundary

Pf: We want to show there exists a nonconstant harmonic function with finite energy.

Notation: i, j, k vertices

z_i, z_j, z_k complex coordinates of vertices = centers

z_{ijk} center of dual circle.

$$c_{ij} := \frac{1}{\sqrt{-1}} \frac{z_{ijk} - z_{jil}}{z_j - z_i} = \frac{|z_{ijk} - z_{jil}|}{|z_j - z_i|}$$

$$> 0$$

$$c_{ij} = c_{ji}$$

Check: $z : V \to \mathbb{C}$ is harmonic with respect to c.

For each i, $$\sum_{j \in V(i)} c_{ij}(z_j - z_i) = \frac{1}{\sqrt{-1}} \sum_{j \in V(i)} \left(z_{ijk} - z_{jil} \right)$$

$$= 0.$$

$x := \mathrm{Re}(z)$. $\qquad y := \mathrm{Im}(z)$.

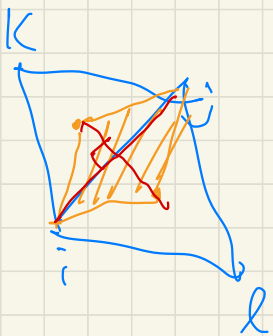
$x : V \to \mathbb{R}$, $\qquad y : V \to \mathbb{R}$

$\Rightarrow$ x, y real-valued harmonic functions with respect to c.

Check: x, y have finite Dirichlet energy.

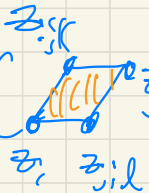
$$D(x) + D(y) = \sum_{ij \in E} c_{ij}\left(|x_j - x_i|^2 + |y_j - y_i|^2\right)$$

$$= \sum_{ij \in E} c_{ij} \, |z_j - z_i|^2$$



$$= \sum_{ij \in E} |z_{ijk} - z_{ijl}| \cdot |z_j - z_i|.$$

$$= \sum_{ij \in E} 2 \, \text{Area}\Big($$



$$\Big)$$

$$= 2 \, \text{Area}(\mathcal{U}) < \infty.$$

$\Rightarrow$ the coordinate functions x, y are

1. harmonic
2. finite Energy
3. bounded.
4. non-constant

(geometric weights).

$\Rightarrow$ G is transient, with respect $c_{ij} := \dfrac{|\text{dual edge}|}{|\text{primal edge}|}$

Claim: $c_{ij} \le 1$.

Pf: $c_{ij} = \dfrac{r_{ijk} + r_{ijl}}{r_i + r_j} = \underbrace{\dfrac{\sqrt{r_i \cdot r_j}}{r_i + r_j}}_{\le \frac{1}{2}} \left(\underbrace{\sqrt{\dfrac{r_k}{r_i + r_j + r_k}}}_{\le 1} + \underbrace{\sqrt{\dfrac{r_l}{r_i + r_j + r_l}}}_{\le 1} \right) \le 1$

$\Rightarrow \dfrac{1}{c_{ij}} \ge 1 \Rightarrow \infty > R_{eff,c}(a, \infty) \ge R_{eff,\, c \equiv 1}(a, \infty)$

$\Rightarrow$ G is transient with respect $c \equiv 1$. (combinatorial weights)

Circle packing Thm

Suppose $G=(V,\vec{E})$ 1-skeleton of triangulation of open disk.

We assume G has bounded vertex circle packing.

(1). $\exists$ circle packing in unit disk $\Leftrightarrow$ G is transient

(2). $\exists$ circle packing filling up $\mathbb{R}^2$ $\Leftrightarrow$ G is recurrent.

Remark No circle packing with combinatorics of a triangle lattice in a bounded domain.

Def $c, \tilde{c}$ two edge weights on a graph are said to be comparable if $\exists\, K > 0$ s.t. $\forall\, xy \in E$,

$$\frac{1}{K}\tilde{c}_{xy} \le c_{xy} \le K\,\tilde{c}_{xy}$$

Exercise: If $c, \tilde{c}$ comparable, then

G is transient with respect to c $\Leftrightarrow$ transient respect to $\tilde{c}$.

Philosophy: geometric weights from circle packings are comparable to combinatorial edge weights,

Royden's Decomposition

Pick $o \in V$,

$$D = \{ u : V \to \mathbb{R} \mid D(u) := \sum_{xy \in E} c_{xy} (u_y - u_x)^2 < \infty \},$$

$$D_0 = \overline{\{ u : V \to \mathbb{R} \mid u \text{ has finite support} \}}.$$

$$u, v \in D \Rightarrow \langle u, v \rangle := u(o) v(o) + \sum_{xy \in E} c_{xy} (u_y - u_x)(v_y - v_x)$$

Denote $e : V \to \mathbb{R}$ as constant function $e \equiv 1$.

Thm (Γ, r). Then the following are equivalent.

1). (Γ, r) is recurrent

2). $e \in D_0$

3). $D = D_0$

Pf: 1) $\Rightarrow$ 2).

For each n: $u_n : V \to \mathbb{R}$

Assume (Γ, r) recurrent,

$$\Rightarrow \quad 0 = \left(R_{eff}(0,\infty)\right)^{-1} = \inf\left(D(u) \;\middle|\; u(0)=1,\ u \text{ has finite support} \right)$$

$\Rightarrow \exists \ \{u_n\}$ s.t. $u_n(0)=1$, $D(u_n) \to 0$ as $n \to \infty$.

Check: $$\|u_n - e\|^2 = \cancel{(u_n(0)-e(0))^2}^{\,0} + D(u_n - e)$$
$$= D(u_n) \longrightarrow 0 \quad \text{as } n \to \infty.$$

$\Rightarrow$ $\{u_n\}$ converges to e with respect to $\|\cdot\|_D$.

$\Rightarrow$ $e \in D_0$

2) $\Rightarrow$ 1) exercise.

3) $\Rightarrow$ 2) is obvious.

Need to show 2) $\Rightarrow$ 3).

Assume $e \in D_0$.

$\Rightarrow \exists \{u_n\}$ with finite support

s.t. $\|u_n - e\| \to 0$ as $n \to \infty$.

Idea: Let $f \in D$.

Want to construct $\{f_n\}$,

s.t. ① f_n has finite support

② $\|f_n - f\|_D \to 0$ as $n \to \infty$.

With out loss of generality, assume,

① u_n non-negative.

② $\|u_n - e\| < \frac{1}{n^2}$

$\Rightarrow$ $v_n := n \cdot u_n$ has finite support,

For each fix x, $\lim_{n\to\infty} v_n(x) = \infty$.

Suppose f is non-negative,

$$f_n(x) := \min\{f(x), v_n(x)\}.$$

If $v_n(x) = 0$, $\Rightarrow$ $f_n(x) = 0$,

$\Rightarrow$ $\operatorname{supp}(f_n) \subset \operatorname{supp}(v_n)$

$\Rightarrow$ f_n has finite support.

Furthermore, $\lim_{n\to\infty} v_n(x) \to \infty$ $\Rightarrow$ $\lim_{n\to\infty} f_n(x) = f(x)$,

exercise: $\lim_{n\to\infty} \|f_n - f\|_D = 0$.

Recurrent: $D = D_0 \cancel{\oplus HD} \to \phi.$

Tomorrow: Transient. $D = D_0 \oplus HD$

harmonic function with finite energy.

"$\dim HD \approx$ # points of boundary"