Lecture 11   18 April, 2022   Recurrent network.

Know: Recurrent network $\Rightarrow$ no noncst. ~~nonnegative~~ superharmonic function

nonpositive subharmonic

Part (I) ✓

Thm Recurrent network $\Rightarrow$ no nonconst. superharmonic (subharmonic) function with finite Dirichlet energy.

Part (II)

Moreover, for any 0-chain $z \in \partial H_1$, the soln is unique.

i.e. $\exists$ unique $I \in H_1$ s.t. $\partial I = z$.

(e.g. $I_M = I_L = I$)

Pf. Let $u \in D$ (finite energy) and $u(x) \geq Pu(x)$ for all x (superharmonic).

Consider $f(x) = u(x) - Pu(x) \geq 0$.

Claim: $f \equiv 0$.

Note: u is the potential of the current I that solves $\partial I + \eta = 0$ where $\eta(x) = c(x) f(x)$.

$\eta \in \partial H_1 \iff W(I_n') = \sum_{x,y \in V_n} c(x) G_n'(x,y) f(x) f(y) < \infty$ as $n \to \infty$.

If $f(x) \neq 0$, $c(x) G_n'(x,x) f(x) f(x) \leq W(I_n')$

$\Rightarrow$ Contradiction since $G_n'(x,x) \to G(x,x) = \infty$ as $n \to \infty$.

$\Rightarrow \quad f \equiv 0.$

$\Rightarrow \quad u$ is harmonic.

Claim: u is constant.

Consider $u = u_+ - u_-$

where. $u_+(x) = \max\{u(x), 0\} \geq 0$

$u_-(x) = \max\{-u(x), 0\} \geq 0.$

Notice that: $u(x) = \sum_{y \in V(x)} p(x,y)\, u(y) \leq \sum_{y \in V(x)} p(x,y)\, u_+(x)$

If $u(x) \geq 0 \Rightarrow u_+(x) = u(x) \leq (P u_+)(x)$

$u(x) < 0 \Rightarrow u_+(x) = 0 \leq (P u_+)(x)$

$\Rightarrow u_+$ subharmonic.

Check: $u_+ \in D.$

$$\sum c(x,y)\,[u_+(x)-u_-(x)]^2 \le \sum c(x,y)\,|u(x)-u(y)|^2 = D(u) < \infty.$$

$\Rightarrow$ $u_+ \in D.$

$\Rightarrow$ u_+ finite energy and subharmonic $\Rightarrow$ u_+ harmonic

$\Rightarrow$ superharmonic. $u_+ \ge 0$

since recurrent $\Rightarrow$ u_+ const.

Similarly, u_- is const.

$\Rightarrow$ u is const.

Part (ii). Assume $z \in \partial H$,

$u, \hat{u}$ are potential of $I, \tilde{I}$ s.t. $\partial I = -z = \partial \tilde{I}.$

$\Rightarrow \quad \partial(I-\tilde{I})=0$

$\Rightarrow \quad u-\tilde{u}$ harmonic

Part (I) $\Rightarrow \quad u-\tilde{u}$ constant.

$\Rightarrow \quad I \equiv \tilde{I}$.

Corollary: If $\iota \in \partial H$, set $f(x) = -c(x)\,\iota(x)$.

Fix any $q \in V$. If $u \in D$ satisfies,

$$(Id-P)u\,(x) = f(x) \qquad \forall\ x \in V-\{q\}.$$

$$\Rightarrow \quad (Id-P)u\,(q) = f(q).$$

Pf: Pick $\tilde{u}$ potential of I that solves $\partial I + \iota = 0$ everywhere.

Consider $u^{+} := \hat{u} - u$.

$$(Id - P)\, u^{+}(x) = \begin{cases} 0 & \text{if } x \neq g. \\ \delta & \text{if } x = g. \end{cases}$$

If $\delta \geq 0$, $\Rightarrow$ u^{+} superharmonic.

$\Rightarrow$ u^{+} const.

$\Rightarrow$ $\delta = 0$.

If $\delta \leq 0$, ... subharmonic $\Rightarrow$ $\delta = 0$.

Eg. $\eta = \delta_a$ & ∂H, if (Γ, r) is recurrent.

Pf:

$\sum_{y \in V(x)} c(x,y)(u(x)-u(y)) = 0 \Leftrightarrow (Id - P)u(x) = 0$ for all $x \neq a$.

unique soln is $u \equiv$ const,

$(Id - P)u(a) = 0.$

Fix $q \in V_1$

$(\Gamma, r) \to \cdots \subset \Gamma_n \subset \Gamma_{n+1} \cdots$ $\to$ $\Gamma_n^q, \Gamma_{n+1}^q, \ldots$

finite networks

$\left(\begin{array}{l} \text{In } \Gamma_n' \\ V_n' = V_n \cup \{b_n\}. \\ b_n \to \infty \text{ as } n \to \infty \end{array} \right)$

For Γ_n^q, b_n

$V_n^q = (V_n - \{q\}) \cup \{q\}$.

$b_n = q$ for all n.

We short all vertices of $V - V_n$ to q.

Given any 0-chain τ,

$$\tau_n(x) = \begin{cases} \tau(x) & \text{if } x \in V_n - \{q\}. \\ \sum_{x \in V_n - \{q\}} -\tau(x) & \text{if } x = q. \end{cases}$$

Important: $\tau_n(q) \neq \tau(q)$

Solve $\partial I_n + \tau_n = 0$ over Γ_n^q.

Potential of $I_n \Rightarrow u(x) := \sum_{y \in V_n^q} G_n(x, y; q) f(y)$

where $f(x) = -c(x)\tau(x)$.

$$W(I_n) = \sum_{x, y \in V_n} c(x) G_n(x, y; q) f(x) f(y)$$

Claim: $\lim_n G_n(x,y;q) = G(x,y;q) < \infty$.

Thm (Γ, r) recurrent. $z \in \partial H_1$.

$\lim_{n\to\infty} W(I_n)$ exists and converge to $W(I)$ where

$\partial I + z = 0$

$\langle I, Z \rangle = 0$ for all finite cycles Z.

Thm Fix any $q \in V$. For z 0-chain

Same if consider infinite cycles

Then, $\exists$ soln I s.t. $\partial I + z = 0$ for $x \in V - \{q\}$.

$\langle I, Z \rangle = 0$ for all finite cycles Z.

$\iff \{W(I_n)\}$ is bounded. where I_n defined on Γ_n^q.

Book: ~~$z \in \partial H_1 \iff W(I_n)$ is bounded~~

Not correct.

Pf: $W(I_n)$ is bounded $\Rightarrow$ $I_n \to I \in H_1$ weakly

$$\partial I_n + z_n = 0 \xRightarrow{\text{weak convergence}} \partial I + z = 0. \text{ for } x \neq q.$$

Another word: Want $z \in \partial H_1$,

and know $z(x)$ for $x \in V - \{q\}$

$\Rightarrow$ $z(q)$ determined.

$$z(q) = -\sum_{x \in V - \{q\}} z(x)$$

Goal on Wed: If $\sum_x |z(x)| < \infty$ and $z \in \partial H_1$

$$\Rightarrow \sum_{x \in V} z(x) = 0$$

Recurrent: $\left(z \in \partial H_1 \overset{?}{\Rightarrow} \sum_x z(x) = 0 \right) \Rightarrow$ doesn't make sense unless we know $\sum_x |z(x)| < \infty$

$\partial I_n + z_n = 0$, $z_n(q) \neq z(q)$.

(Value of $z_n(q)$ doesn't matter, in constructing potential) $\Rightarrow u_n(x) := \sum G_n(x,y;q)\, c(y)\, z_n(y)$.

Note: $G_n(x,q;q) = 0$.

$$\partial : \underset{\substack{\cup \\ I}}{H_1} \longrightarrow \underset{\substack{\cup \\ z}}{H_0}$$

means $\sum_{xy \in E} r(xy) I^2(x,y) < \infty$ $\Leftrightarrow$ $\sum_{x \in V} \frac{|z(x)|}{c(x)} < \infty$.