Prop 4.3 Let $p: Z \to \mathbb{A}^1_F$ be a finite surjective morphism and suppose that Z is integral. Let $f_1, \dots, f_n \in \mathcal{O}^*(Z)$ and

$$p^*(0) = \sum n_i^0 z_i^0 \ , \quad p^*(1) = \sum n_i^1 z_i^1$$

in $Z_0(Z)$. Define $\forall t = 0, 1$

$$\varphi_t = \sum_i n_i^t \cdot N_{k(z_i^t)/F}\left([f_1] \cdots [f_n]\big|_{z_i^t} \right).$$

Then $\varphi_0 = \varphi_1 \in K_n^M(F)$.

Proof: The $k(Z)/F$ is an algebraic function field. Suppose $x = [\frac{t}{t-1}][f_1] \cdots [f_n]$ and $v \in DV(k(Z)/F)$. If $v|_{k(\mathbb{A}^1_F)} = v_\infty$, $\frac{t}{t-1} = 1$ if $t = \infty$, so $\partial_v(x) = 0$. Otherwise if $v|_{k(\mathbb{A}^1_F)} \neq v_1, v_0$, the $\frac{t}{t-1}, f_1, \dots, f_n$ have no valuation at v.

If $v|_{k(\mathbb{A}^1_F)} = v_0$, then v is centered at some z_i^0. $\tilde{Z} \xrightarrow{p} Z$ (normalization)

$$\sum_{p(v) = z_i^0} N_{k(v)/k(z_i^0)}(\partial_v(x)) = n_i^0 [f_1] \cdots [f_n]\big|_{z_i^0} \qquad \frac{t}{t-1} = 0, \ t = 0$$

Similarly, $\sum_{p(v) = z_i^1} N_{k(v)/k(z_i^1)}(\partial_v(x)) = -n_i^1 [f_1] \cdots [f_n]\big|_{z_i^1}$ $\quad v_1(\frac{t}{t-1}) = -1$

By Weil Reciprocity (Thm 3.12), we find that $\sum N_{k(v)/F} \partial_v(x) = \varphi_0 - \varphi_1 = 0$. □

Now we define a map $\theta: H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \to K_n^M(F)$ as following: $\quad \operatorname{coker}\left(\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1) \xrightarrow{\partial_1 - \partial_0} \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F) \right)$

Every closed point x of $(\mathbb{G}_m^{\times n})_F$ corresponds to $(x_1, \dots, x_n)$, $x_i \in k(x)^\times$.

Then define $f: \mathbb{Z}(\mathbb{G}_m^{\wedge n})(\operatorname{Spec} F) \longrightarrow K_n^M(F)$ $\quad$ [$= \mathbb{Z}(\text{closed pts in } \mathbb{G}_m^{\times n}) / \{(x_1, \dots, 1, \dots, x_n)\}$]

$$x \longmapsto N_{k(x)/F}([x_1] \cdots [x_n])$$

If one of $x_i = 1$, $f(x) = 0$ so it's well-defined.

The $\mathbb{Z}(\mathbb{G}_m^{\wedge n})(\mathbb{A}^1)$ is generated by $C \subseteq \mathbb{A}^1 \times \mathbb{G}_m^{\times n}$ (finite surj. $\to \mathbb{A}^1$). Such C gives $f_1, \dots, f_n \in \mathcal{O}^*(C)$.

Then by Prop 4.3, we have $f \circ (\partial_0 - \partial_1) = 0$. Hence the θ is defined.

Conversely, every $x \in F^\times$ corresponds to a map $x: \operatorname{Spec} F \to \mathbb{G}_m$, which gives $\lambda_F([x]) \in H^{1,1}(\operatorname{Spec} F, \mathbb{Z})$.

$$\lambda_F: K_*^M(F) \to H^{*,*}(\operatorname{Spec} F, \mathbb{Z}), \quad K_*^M(F) = T(F^\times)/a \otimes 1-a, \ a \in F^\times \setminus 1, \quad x \mapsto$$

To prove λ_F is well defined, we are going to show $\lambda_F([x]) \cdot \lambda_F([1-x]) = 0$ if $x \neq 0, 1$.

$$\lambda_F([x]) \cdot \lambda_F([1-x]) = [x : 1-x] \in H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$$

$$[ab : c] = [a : c] + [b : c].$$

Prop 4.4 Suppose $\exists n > 0$ s.t. $n[x : 1-x]$ for all finite extensions of F and all $x \neq 0, 1$. Then $[x : 1-x] = 0$ for all $x \neq 0, 1$.

Proof: Suppose $n = m \cdot p$ where p is a prime. We claim $m[x : 1-x] = 0$. Let us consider $y = \sqrt[p]{x}$ ($y \notin F$) and $E = F(y)$. Then

$$0 = mp[y : 1-y] = m[x : 1-y], \quad 1-x = N_{E/F}(1-y).$$

Hence $\quad 0 = N_{E/F}(m[x : 1-y]) = m[x : N_{E/F}(1-y)] = m[x : 1-x]$

If F is a splitting field of $T^p - x \in F[T]$, $\{y_i\} = \{p\text{-th roots of } x\}$, $\sum_i m[x : 1-y_i] = m[x : 1-x]$. □ $\quad x = y_i^p, \ \sum mp[y_i : 1-y_i] = 0$

Prop 4.5 $[x : 1-x] = 0 \ (x \neq 0, 1)$.

Proof: Let $Z \in \operatorname{Cor}_k(\mathbb{A}^1, \mathbb{G}_m)$ be the cycle in $\mathbb{A}^1 \times \mathbb{G}_m = (t, a)$ defined by

$$a^3 - t(x^3+1)a^2 + t(x^3+1)a - x^3 = 0.$$

Let w be a root of $a^2 + a + 1$, so $w^3 = 1$. Suppose $E = F(w)$. The fiber of Z over $t = 0$ are $\{a = x\}, \{a = wx\}, \{a = w^2 x\}$; fiber over $t = 1$ are $\{a = x^3\}$, $\{a = -w\}$ and $\{a = -w^2\}$.

Suppose $x^3 \neq 1$. Then Z comes from $\operatorname{Cor}(\mathbb{A}^1, \mathbb{G}_m \setminus \{1\})$ and $Z \cong \mathbb{G}_m \setminus \{1\}$. Composing Z with the map $\mathbb{G}_m \setminus \{1\} \to \mathbb{G}_m^{\times 2}$, we obtain a $Z' \in \operatorname{Cor}(\mathbb{A}^1, \mathbb{G}_m^{\times 2})$.

$$\mathbb{A}^1 \xrightarrow{Z} \mathbb{G}_m \setminus \{1\} \to \mathbb{G}_m^{\times 2}, \quad a \mapsto (a, 1-a), \quad \text{composite } Z'$$

Then in $H^{2,2}(\operatorname{Spec} F, \mathbb{Z})$, we have

$$\partial_0(Z') = [x : 1-x] + [wx : 1-wx] + [w^2 x : 1-w^2 x]$$

$$\partial_1(Z') = [x^3 : 1-x^3] + [-w : 1+w] + [-w^2 : 1+w^2].$$

$[wx : 1-wx] = [x : 1-wx] + [w : 1-wx]$, $[w^2 x : 1-w^2 x] = [w^2 : 1-w^2 x] + [x : 1-w^2 x]$
$= [w : (1-w^2 x)^2] + [x : 1-w^2 x]$

$$\partial_0(Z') = [x : 1-x] + [x : 1-wx] + [w : 1-wx] + [w : (1-w^2 x)^2] + [x : 1-w^2 x]$$

$$= [x : 1-x^3] + [w : (1-wx)(1-w^2 x)^2].$$

$3\partial_0(Z') = [x^3 : 1-x^3] = 3\partial_1(Z') = 3[x^3 : 1-x^3]$. So $2[x^3 : 1-x^3] = 0$ over E. Then $0 = N_{E/F}(2[x^3 : 1-x^3]) = 2[N_{E/F}(x^3) : 1-x^3] = 2[x^6 : 1-x^3] = 4[x^3 : 1-x^3]$ over F.

So if $x = y^3$ in F, we have $4[x : 1-x] = 0$. Otherwise let $K = F(\sqrt[3]{x})$. Then $N(1 - \sqrt[3]{x}) = 1 - x$. So $4[x : 1-x] = 0$ over K. Hence

$$N_{K/F}(4[x : 1-x]) = 4[N_{K/F}(x) : 1-x] = 12[x : 1-x] = 0.$$

If $x^3 = 1$, then $3[x : 1-x] = 0$. So the statement follows from Prop 4.4. □

To show that λ_F is an isomorphism, since $\theta \circ \lambda_F = \mathrm{id}$, it suffices to show that λ_F is surjective.

Prop 4.6 For any finite extension E/F, the diagram

$$\begin{array}{ccc} K_n^M(E) & \xrightarrow{\lambda_E} & H^{n,n}(\operatorname{Spec} E, \mathbb{Z}) \\ \downarrow N_{E/F} & & \downarrow N_{E/F} \\ K_n^M(F) & \xrightarrow{\lambda_F} & H^{n,n}(\operatorname{Spec} F, \mathbb{Z}) \end{array} \qquad \begin{array}{c} y \\ \downarrow \\ x \end{array}$$

commutes.

Proof: Assume that F has no extensions of degree prime to l and that $[E : F] = l$, where l is a prime. By the statement in Prop 3.10 we know that $K_n^M(E)$ is generated by $[f_1] \cdots [f_n]$ where $f_1 \in E$, $f_2, \dots, f_n \in F$. Then statement follows from projection formulas in Prop 3.11 and Prop 4.2

We proved in Thm 3.7 that if $[E : F]$ is a power of l (no finite extension pt degree), then, we could find $E = E_1 \supseteq \cdots \supseteq E_n = F$ s.t. $[E_i : E_{i+1}] = l$, E_i / E_{i+1} is normal.

For general F, by Prop 3.8, the map $K_n^M(F)_{(l)} \to K_n^M(L)_{(l)}$, $H^{n,n}(\operatorname{Spec} F, \mathbb{Z})_{(l)} \to H^{n,n}(\operatorname{Spec} L, \mathbb{Z})_{(l)}$ are injective for some algebraic extension L/F s.t. $\forall$ finite of L has degree a power of l. Moreover, we may assume E/F is a simple extension of prime degree. Hence it's either separable or purely inseparable. In both cases, we could apply Prop 3.9 so we return to the case before □.

Thm 4.7 The map $\lambda_F : K_n^M(F) \to H^{n,n}(\operatorname{Spec} F, \mathbb{Z})$ is an isomorphism of rings.

Proof: If $x \in (\mathbb{G}_m^{\times n})_F$ is a rational point, it's in $\operatorname{Im}(\lambda_F)$ by construction. In general, $x \in \mathbb{G}_m^{\times n}$ (closed) is the push-forward of a rational point of $(\mathbb{G}_m^{\times n})_{k(x)}$. $\quad (\mathbb{G}_m^{\times n})_{k(x)} \to (\mathbb{G}_m^{\times n})_F$

So the statement follows from 4.6. □

§5 Categories of Effective motives

Lem 5.1: Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a functor between small categories and $\mathcal{M}$ be a category with arbitrary colimits. Then the functor

$$\varphi_* : \operatorname{PreShv}(\mathcal{D}, \mathcal{M}) \to \operatorname{PreShv}(\mathcal{C}, \mathcal{M})$$
$$F \longmapsto F \circ \varphi$$

has a left adjoint φ^*.

Proof. Suppose that $G \in \operatorname{PreShv}(\mathcal{C}, \mathcal{M})$. For every $Y \in \mathcal{D}$, define C_Y to be the category $\begin{cases} \text{Obj}: Y \to \varphi(X), \ X \in \mathcal{C} \\ \text{Mor}: Y \to \varphi(X_1) \xrightarrow{\varphi(b)} \varphi(X_2), \ b: X_1 \to X_2 \end{cases}$. We have a contravariant functor

$$\theta_Y : C_Y \longrightarrow \mathcal{M}$$
$$Y \to \varphi(X) \longmapsto GX.$$

Then define $(\varphi^* G)Y = \varinjlim \theta_Y$. For any morphism $c: Y_1 \to Y_2$ in $\mathcal{D}$, we define $\varphi^*(G)(c)$ by the diagram

$$\begin{array}{ccc} \theta_{Y_2}(Y_2 \to \varphi(X)) & \to & \theta_{Y_1}(Y_1 \to \varphi(X)) \\ \downarrow & & \downarrow \\ \varinjlim \theta_{Y_2} & \xrightarrow{\varphi^*(G)(c)} & \varinjlim \theta_{Y_1} \\ \parallel & & \parallel \\ (\varphi^* G) Y_2 & & (\varphi^* G)(Y_1) \end{array}$$

. □