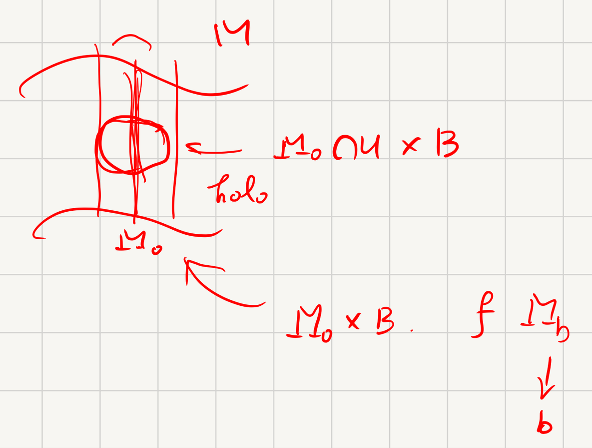
§ Transversely holomorphic trivializations

- $\mathcal{B}^{**}_{M}$ sheaf
- $M_b \xrightarrow[\sim]{\text{diff}} M_{b'}$



- Definition. A transversally holomorphic trivialization of a deformation $M_0 \xrightarrow{i} M \xrightarrow{f} (B,0)$ is a diffeomorphism (holomorphism?)

$$\begin{tikzcd} M_0 \times \Delta \arrow[r, "\phi"] \arrow[dr, "pr_\Delta"'] & f^{-1}(\Delta) \subseteq M \arrow[d, "f"] \\ & \Delta \end{tikzcd}$$

for some neighbourhood $0 \in \Delta \subseteq B$ such that

(I). $\phi(x, 0) = i(x)$ and $f\phi$ is the projection on the second factor.

(II) For every $x \in M_0$, $\phi: \{x\} \times \Delta \to M$ is a holomorphic function.

- Thm IV.3.1 Every deformation of a compact complex manifold admits a transversely holomorphic trivialization

Proof Notation $f: M \to B$ deformation. $B \subseteq \mathbb{C}^n$ polydisk.

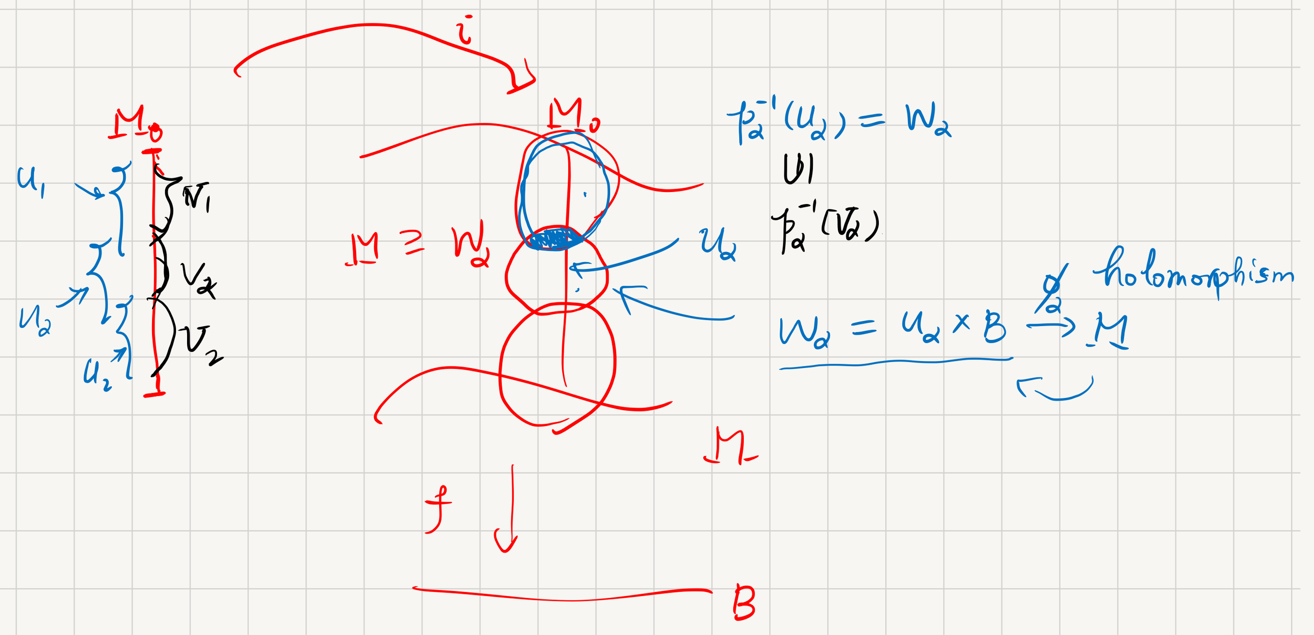
with coordinates $t_1, \cdots, t_n$, and $0 \in B$ base point.

We identify M_0 with $f^{-1}(0)$.

After a possible shrinking of B, we assume there exist a finite open covering $M = \bigcup_\alpha W_\alpha$ and holomorphic projection

$p_\alpha : W_\alpha \to U_\alpha = W_\alpha \cap M_0$ such that

$(p_\alpha, f) : W_\alpha \to U_\alpha \times B$ is a biholomorphism.



Every α, U_α is a local chart of M_0 with coordinates $z_1^\alpha, \cdots, z_m^\alpha : U_\alpha \to \mathbb{C}$.

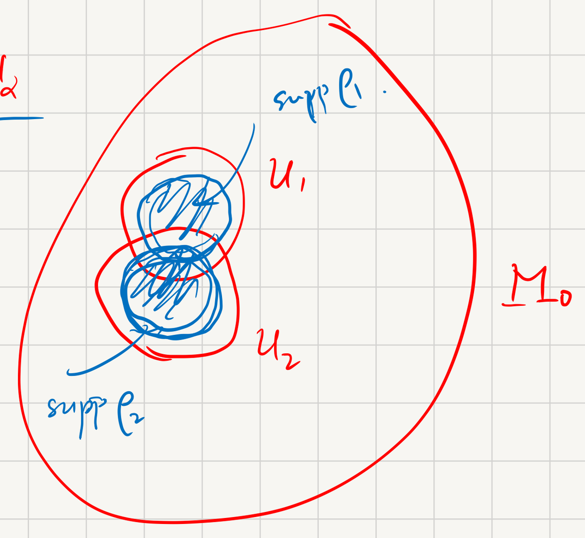
Let $\rho_\alpha : M_0 \to [0,1]$ be a C^∞-partition of unity.

$\{U_\alpha\} \longrightarrow \{V_\alpha\}$ $V_\alpha \subseteq U_\alpha$

Step 1. subordinate to the covering $\{U_\alpha\}$, $\operatorname{supp} \rho_\alpha \subseteq U_\alpha$.

Denote by $V_\alpha = \rho_\alpha^{-1}((0,1])$.

We note that $\{V_\alpha\}$ is a covering of M_0, and $\overline{V_\alpha} \subseteq U_\alpha$



- By assumption, we may assume $p_\alpha^{-1}(\overline{V_\alpha})$ closed in M.

- Step 2. For every subset $C \subseteq \{\alpha\}$, and every $x \in M_0$, we denote

$$H_C = \Big(\bigcap_{\alpha \in C} W_\alpha - \bigcup_{\alpha \notin C} p_\alpha^{-1}(\overline{V_\alpha})\Big) \times \Big(\bigcap_{\alpha \in C} U_\alpha - \bigcup_{\alpha \notin C} \overline{V_\alpha}\Big) \subseteq M \times M_0$$

$$C_x := \{\alpha \mid x \in \overline{V_\alpha}\}$$

$$H := \bigcup_C H_C \subseteq M \times M_0$$

- Remark I. $(x, x) \in H_{C_x}$ for every $x \in M_0$. Then H is an open subset of $M \times M_0$ containing the image $G := \operatorname{im}(M_0 \to M \times M_0)$.
$x \mapsto (x, x)$.

- We wish to find submanifold of $\underset{m+n+m}{M \times M_0}$ which is diffeomorphic to $\underset{m+n}{M}$

$$\underset{m+n}{\widetilde{M}} \subseteq \underset{2m+n}{H} \supseteq \underset{m}{G} = \{(x,x)\} \qquad \underset{m}{M_0} \subseteq M$$

- Remark II. Since the projection $pr : M \times M_0 \to M$ is open, and M_0 is compact, after a possible shrinking of B, we may assume $pr_M(H) = M$.

- Remark III, if $(y, x) \in H$, and $x \in \overline{V_\alpha}$, then $(y, x) \in H_C$ for some C containing α and therefore $y \in W_\alpha$

- Step 3 For every α, consider the C^∞-function $q_\alpha : H \cap (M \times U_\alpha) \to \mathbb{C}^m$

$$q_\alpha(y, x) = \sum_\beta \rho_\beta \frac{\partial z^\alpha}{\partial z^\beta}(x) \big(z^\beta(p_\beta(y)) - z^\beta(x) \big)$$

($p_\beta(y) \in M_0$)

By the property of H, q_α is well defined ane separately holomorphic in the variable y.

If $(y, x) \in H \cap (M \times U_\alpha \cap U_\beta)$, then ($U_\alpha \cap U_\beta \subseteq M_0$)

$$q_\gamma(y, x) = \frac{\partial z^\gamma}{\partial z^\alpha}(x)\, q_\alpha(y, x).$$

and then

$$\Gamma := \{ (y, x) \in H \mid q_\alpha(y, x) = 0 \text{ whenever } x \in U_\alpha \}$$

is a well-defined closed subset of H.

Note that Γ is NOT nessary connected, we need to find a component.

- If $y \in V_\alpha \subseteq M_0$, and x is sufficiently near to y, then

$y \in \bigcap_{\beta \in}$